# Sequence-Space Jacobians II

## The Fake News Algorithm, DAGs, and HANK

ECO2101 Lecture 9
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# Introduction

**Last lecture:** Auclert, Bardóczy, Rognlie, and Straub (2021), Section 2

- KS model as $H(\mathbf{K}, \mathbf{Z}) = 0$ in the sequence space
- IRFs via $d\mathbf{K} = G\, d\mathbf{Z}$ where $G = -H_K^{-1} H_Z$
- Certainty equivalence: perfect foresight = stochastic (to first order)
- SSJ vs. KS, BKM, Reiter

**This lecture:** Sections 3–4 and application

- How to *compute* the Jacobians: the Fake News Algorithm
- General equilibrium via DAGs
- Application: one-asset HANK

**Open question from last time:** How do we compute the Jacobians of the $K$ function w.r.t. $r$ and $w$?

# Computing Jacobians for Heterogeneous-Agent Problems

# General Model Representation

Generalize the $\mathcal{K}$ function: map inputs $\mathbf{X}_t$ ($n_x \times 1$) to outputs $\mathbf{Y}_t$ ($n_y \times 1$) Distribution $D_t$: $n_g \times 1$ vector. Outputs: $Y_t = y_t' D_t$

Three functions $v$, $\Lambda$, $y$, given $D_0$:

$$\mathbf{v}_t = v(\mathbf{v}_{t+1}, \mathbf{X}_t) \tag{10}$$

$$D_{t+1} = \Lambda(\mathbf{v}_{t+1}, \mathbf{X}_t)' \, \mathbf{D}_t \tag{11}$$

$$Y_t = y(v_{t+1}, X_t)' \, \mathbf{D}_t \tag{12}$$

- (10): value function — backward iteration
- (11): distribution — forward iteration, $\Lambda$ is $n_g \times n_g$ transition matrix
- (12): aggregation — dot product of individual outcomes $y_t$ with $D_t$

These define a mapping:

$$\mathbf{Y} = h(\mathbf{X})$$

**Goal:** Jacobian $\mathcal{J}$ of $h$ at $\mathbf{X}_{ss}$, at the fixed point when $\mathbf{X}_t = \mathbf{X}_{ss}\ \forall t$.

# Example: Krusell–Smith in This Framework

Mapping the objects:
$X_t = (r_t, w_t)$, $Y_t = (K_t, C_t)$

**Backward (10):** Euler equation maps $v_{t+1}, X_t \to$ policies $k_t, c_t$; envelope:
$v_t = (1 + r_t)u'(c_t)$

**Forward (11):** $k_t$ + exogenous $e$ process $\Rightarrow$ transition matrix $\Lambda_t'$ via Young (2010) lottery

**Aggregation (12):** $K_t = k_t' D_t$, $C_t = c_t' D_t$ with $y_t \equiv (k_t, c_t)$

This backward–forward–aggregation structure is shared by a large class of HA models: one-asset HANK, two-asset HANK, heterogeneous firms, …

# What Is $\mathcal{J}$ and How Does It Relate to $G$?

**Last lecture**: the GE impulse response matrix

$$G = -H_K^{-1} H_Z \qquad \text{maps } d\mathbf{Z} \to d\mathbf{K}$$

**This lecture**: the partial equilibrium Jacobian $\mathcal{J}$ of the HA block

For example, $\mathcal{J}^{K,r}$ has $(t, s)$ entry $\frac{\partial K_t}{\partial r_s}$: the response of aggregate capital at date $t$ to a perturbation of the interest rate at date $s$, holding all other inputs fixed

**The connection** (recall equation (9) from last lecture):

$$[H_K]_{t,s} = \underbrace{\frac{\partial K_t}{\partial r_{s+1}}}_{\text{from } \mathcal{J}^{K,r}} \frac{\partial r_{s+1}}{\partial K_s} + \underbrace{\frac{\partial K_t}{\partial w_{s+1}}}_{\text{from } \mathcal{J}^{K,w}} \frac{\partial w_{s+1}}{\partial K_s} - \mathbf{1}\{s = t\}$$

**The pipeline**: $\underbrace{\mathcal{J}^{K,r}, \mathcal{J}^{K,w}}_{\text{Fake News Alg.}} \xrightarrow{\text{chain rule / DAG}} \underbrace{H_K, H_Z}_{\text{GE Jacobians}} \xrightarrow{\text{invert}} \underbrace{G = -H_K^{-1} H_Z}_{\text{GE IRFs}}$

# Notation

- Let $n_x = n_y = 1$ for now.
- Let $\mathbf{e}_s$ be $T \times 1$ be have 0's everywhere except at the sth entry, where it has a 1.
- Shock at time $s$: $\mathbf{X}^s = \mathbf{X}_{ss} + \mathbf{e}_s \, dx$ for a given $dx$.
- Solutions to (10)-(12) given $\mathbf{X}^s$: $v_t^s$, $D_t^s$, $Y_t^s$, $\Lambda_t^s \equiv \Lambda(v_{t+1}^s, X_t^s)$, $y_t^s \equiv y(v_{t+1}^s, X_t^s)$
- Deviations: $dY_t^s \equiv Y_t^s - Y_{ss}$, etc. From (12) and (11):

$$Y_t^s = (y_t^s)' D_t^s \tag{14}$$

$$D_{t+1}^s = (\Lambda_t^s)' D_t^s \tag{15}$$

Column $s$ of Jacobian $\mathcal{J}$: the limit of $d\mathbf{Y}^s/dx$ as $dx \to 0$.

# The Jacobian $\mathcal{J}$: What Does It Look Like?

The mapping $\mathbf{Y} = h(\mathbf{X})$ takes input sequences to output sequences

Its Jacobian $\mathcal{J}^{Y,X}$ is a $T \times T$ matrix:

$$\underset{T\times T}{\mathcal{J}^{Y,X}} = \begin{array}{c|c} & \begin{matrix} X_0 & X_1 & \cdots & X_{T-1} \end{matrix} \\ \hline \begin{matrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{T-1} \end{matrix} & \begin{bmatrix} \frac{\partial Y_0}{\partial X_0} & \frac{\partial Y_0}{\partial X_1} & \cdots & \frac{\partial Y_0}{\partial X_{T-1}} \\ \frac{\partial Y_1}{\partial X_0} & \frac{\partial Y_1}{\partial X_1} & \cdots & \frac{\partial Y_1}{\partial X_{T-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Y_{T-1}}{\partial X_0} & \frac{\partial Y_{T-1}}{\partial X_1} & \cdots & \frac{\partial Y_{T-1}}{\partial X_{T-1}} \end{bmatrix} \end{array}$$

- **Column** $s$: how does perturbing input $X_s$ affect the *entire* output path $\{Y_t\}$?
- **Row** $t$: how sensitive is $Y_t$ to inputs at *every* date?
- With $n_x$ inputs and $n_y$ outputs: $n_x \times n_y$ such matrices

# Jacobian $\mathcal{J}$ for the KS example

$$\underset{T\times T}{\mathcal{J}^{K,r}} = \begin{array}{c|cccc} & r_0 & r_1 & \cdots & r_{T-1} \\ \hline K_0 & \frac{\partial K_0}{\partial r_0} & \frac{\partial K_0}{\partial r_1} & \cdots & \frac{\partial K_0}{\partial r_{T-1}} \\ K_1 & \frac{\partial K_1}{\partial r_0} & \frac{\partial K_1}{\partial r_1} & \cdots & \frac{\partial K_1}{\partial r_{T-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{T-1} & \frac{\partial K_{T-1}}{\partial r_0} & \frac{\partial K_{T-1}}{\partial r_1} & \cdots & \frac{\partial K_{T-1}}{\partial r_{T-1}} \end{array}$$

- **Column** $s$: response of $\{K_t\}$ to a unit perturbation of $r_s$, *holding* $w$ *and all other* $r_{s'}$ *fixed*
- **Partial equilibrium**: prices do not adjust, no market clearing imposed
- In KS we need two such matrices: $\mathcal{J}^{K,r}$ and $\mathcal{J}^{K,w}$

# The Direct Method

**Direct method** for column $s$ of $\mathcal{J}$:

1. Shock $dx$ at time $s$.
2. Iterate (10) backward from $v_T = v_{ss}$
3. Iterate (11) forward from $D_0 = D_{ss}$
4. Aggregate: $Y_t^s = (y_t^s)' D_t^s$, form $\mathcal{J}_{t,s} = dY_t^s / dx$

Repeat $T$ times $\Rightarrow$ cost: $n_x T^2$ backward + $n_x T^2$ forward steps

With $T = 300$: $\sim 90{,}000$ backward iterations per input. Too slow.

Can we do better? **Yes** — the columns of $\mathcal{J}$ are closely related to each other

# Direct Method vs. BKM: What Are We Computing?

**Direct method**: builds columns of $\mathcal{J}$ – the PE Jacobian

- Perturb input $r_s$ by $dx$, *holding all other inputs fixed*
- One backward pass + one forward pass $\Rightarrow$ column $s$ of $\mathcal{J}^{K,r}$
- Repeat $T$ times $\Rightarrow$ cost $O(T^2)$
- Fake news replaces this: same $\mathcal{J}$, cost $O(T)$

**BKM** (2018): builds columns of $G$ – the GE impulse response matrix

- Perturb $Z_s$ by $dZ$, then solve for the *full nonlinear GE transition*
- Each column requires iterating over guesses for $\{K_t\}$ until markets clear
- Much more expensive per column, no convergence guarantee

**The SSJ strategy**: compute $\mathcal{J}$'s cheaply (fake news), then get $G$ via the chain rule

$$\underbrace{\mathcal{J}^{K,r},\ \mathcal{J}^{K,w}}_{\text{fake news: } O(T)} \xrightarrow{\text{DAG / chain rule: ms}} H_K,\ H_Z \xrightarrow{\text{invert}} G$$

# The Fake News Algorithm

# Key Idea 1: Policy Function Symmetry

**Lemma 1**

For $s \geq 1$, $t \geq 1$:

$$y_t^s = \begin{cases} y_{ss} & s < t \\ y_{T-1-(s-t)}^{T-1} & s \geq t \end{cases} \qquad \Lambda_t^s = \begin{cases} \Lambda_{ss} & s < t \\ \Lambda_{T-1-(s-t)}^{T-1} & s \geq t \end{cases} \tag{16}$$

**Intuition**: agents care about distance to the shock $s - t$, not calendar time

**Consequence**: a *single* backward iteration from $s = T - 1$ gives all $y_t^s$ and $\Lambda_t^s$

From this single pass we extract two objects:

- $\mathcal{Y}_s\, dx \equiv dY_0^s = (dy_0^s)' D_{ss}$: impact effect on output at date 0 (first row of $\mathcal{J}$)
- $\mathcal{D}_s\, dx \equiv dD_1^s = (d\Lambda_0^s)' D_{ss}$: perturbation to date-1 distribution

# Key Idea 2: The Fake News Matrix

For $s, t \geq 1$, define:

$$\mathcal{F}_{t,s} \cdot dx \;\equiv\; dY_t^s - dY_{t-1}^{s-1} \tag{17}$$

Since $dy_t^s = dy_{t-1}^{s-1}$ from (16), one might expect $\mathcal{F}_{t,s} = 0$. But the distribution changes!

**Lemma 2**

For $D_0 = D_{ss}$, infinitesimal $dx$, $s, t \geq 1$:

$$\mathcal{F}_{t,s} \cdot dx = \mathbf{y}_{ss}'(\Lambda_{ss}')^{t-1} d\mathbf{D}_1^s \tag{18}$$

**Sketch of proof**: Linearize (14): $dY_t^s = y_{ss}' dD_t^s + (dy_t^s)' D_{ss}$. Subtract adjacent columns, use $dy_t^s = dy_{t-1}^{s-1}$ and $d\Lambda_{t-1}^s = d\Lambda_{t-2}^{s-1}$ to show $dD_t^s - dD_{t-1}^{s-1} = (\Lambda_{ss}')^{t-1} dD_1^s$

# Intuition: Fake News Shocks

- $\mathcal{F}_{t,s} \cdot dx \equiv dY_t^s - dY_{t-1}^{s-1}$: what's *different* about the response to a shock at $s$ vs. a shock at $s-1$ (shifted by one period)?
- From Lemma 1: *policies* are the same (only distance to shock matters). The *only* difference: at $t = 0$, agents anticipated the date-$s$ shock and adjusted their behavior, perturbing the distribution to $D_1^s \neq D_{ss}$
- **"Fake news" thought experiment** for $F_{t,s}$:
  - $\rightarrow$ $t = 0$: agents hear "shock will hit at date $s$" $\Rightarrow$ they react, distribution moves to $D_1^s$
  - $\rightarrow$ $t = 1$: announcement is retracted $\Rightarrow$ agents revert to steady-state policies
  - $\rightarrow$ $t \geq 1$: the only remaining effect is the distribution perturbation $dD_1^s$, propagating forward under SS dynamics: $\mathcal{F}_{t,s} \cdot dx = \mathcal{E}'_{t-1}\, dD_1^s$
- **Key insight**: a *news shock* (column $s$ of $\mathcal{J}$) = accumulation of *fake news shocks*:

$$\mathcal{J}_{t,s} = \underbrace{\mathcal{F}_{t,s}}_{\text{fake news}} + \underbrace{\mathcal{F}_{t-1,s-1}}_{\text{fake news}} + \cdots + \underbrace{\mathcal{F}_{t-s,0}}_{\text{first column}}$$

# Key Idea 3: Expectation Vectors

**Definition 1**

The **expectation vector** for outcome $Y$ at time $t$:

$$\mathcal{E}_t \equiv (\Lambda_{ss})^t \, y_{ss} \tag{23}$$

$\mathcal{E}_t$: expected outcome $Y$ in $t$ periods, for an agent starting at each grid point

Easy to compute: $\mathcal{E}_0 = y_{ss}, \;\; \mathcal{E}_t = \Lambda_{ss}\mathcal{E}_{t-1}$

Now equation (18) reads: $\mathcal{F}_{t,s} \cdot dx = \mathcal{E}'_{t-1} \, dD_1^s$

$\Rightarrow$ All rows $t \geq 1$ of $\mathcal{F}$ are products of expectation vectors $\mathcal{E}_{t-1}$ and distribution perturbations $\mathcal{D}_s$

# Proposition 1: From $F$ to $\mathcal{J}$

**Proposition 1**

Define:

$$\mathcal{F}_{t,s} \cdot dx \equiv \begin{cases} dY_0^s & t = 0 \\ \mathcal{E}'_{t-1}\, dD_1^s & t \geq 1 \end{cases} \tag{24}$$

where $dY_0^s = (dy_0^s)' D_{ss}$, $dD_1^s = (d\Lambda_0^s)' D_{ss}$. Then:

$$\mathcal{J}_{t,s} = \sum_{k=0}^{\min\{s,t\}} F_{t-k,s-k} \tag{25}$$

with recursion $\mathcal{J}_{t,s} = \mathcal{J}_{t-1,s-1} + F_{t,s}$ for $t, s \geq 1$

Each entry of $\mathcal{J}$ = cumulative sum along the upper-left diagonal of $F$

Example: $\mathcal{J}_{2,2} = \mathcal{E}'_1 \mathcal{D}_2 + \mathcal{E}'_0 \mathcal{D}_1 + \mathcal{Y}_0$

# Building $\mathcal{J}$: Step 1 — The Ingredients

From a single backward pass (shock at $s = T-1$) + steady-state objects, we obtain:

**Impact effects $\mathcal{Y}_s$**

$$\mathcal{Y}_s\, dx = (dy_0^s)' D_{ss}$$

$T$ scalars, $s = 0, \ldots, T-1$

First row of $\mathcal{J}$: how output at $t = 0$ responds to shock at each date $s$

**Distribution shifts $\mathcal{D}_s$**

$$\mathcal{D}_s\, dx = (d\Lambda_0^s)' D_{ss}$$

$T$ vectors ($n_g \times 1$ each), $s = 0, \ldots, T-1$

How shock at date $s$ perturbs the date-1 distribution

**Expectation vectors $\mathcal{E}_t$**

$$\mathcal{E}_0 = y_{ss}, \mathcal{E}_t = \Lambda_{ss}\mathcal{E}_{t-1}$$

$T-1$ vectors ($n_g \times 1$ each), $t = 0, \ldots, T-2$

Expected outcome in $t$ periods from each grid point

These three objects are *all we need* to construct $F$ and then $\mathcal{J}$

# Building $\mathcal{J}$: Step 2 — Assemble the Fake News Matrix $F$

$\mathcal{F}$ is $T \times T$. Rows: response dates $t = 0, \ldots, T-1$. Columns: shock dates $s = 0, \ldots, T-1$

$$\underset{T\times T}{\mathcal{F}} = \begin{array}{c} \\ t{=}0 \\ t{=}1 \\ t{=}2 \\ t{=}3 \\ \vdots \end{array} \!\! \begin{array}{c} \begin{array}{cccc} s{=}0 & s{=}1 & s{=}2 & \cdots \end{array} \\ \left[\begin{array}{cccc} \mathcal{Y}_0 & \mathcal{Y}_1 & \mathcal{Y}_2 & \cdots \\ \mathcal{E}_0{}'\mathcal{D}_0 & \mathcal{E}_0{}'\mathcal{D}_1 & \mathcal{E}_0{}'\mathcal{D}_2 & \cdots \\ \mathcal{E}_1{}'\mathcal{D}_0 & \mathcal{E}_1{}'\mathcal{D}_1 & \mathcal{E}_1{}'\mathcal{D}_2 & \cdots \\ \mathcal{E}_2{}'\mathcal{D}_0 & \mathcal{E}_2{}'\mathcal{D}_1 & \mathcal{E}_2{}'\mathcal{D}_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right] \end{array}$$

- First row: impact effects $\mathcal{Y}_s$ (from backward pass)
- Rows $t \geq 1$: dot product of $\mathcal{E}_{t-1}$ and $\mathcal{D}_s$ — each entry is a scalar

# Building $\mathcal{J}$: Step 3 — Accumulate Diagonals

$\mathcal{J}$ is $T \times T$. Rows: response dates $t$. Columns: shock dates $s$. Rule: $\mathcal{J}_{t,s} = \mathcal{J}_{t-1,s-1} + F_{t,s}$

$$\underset{T\times T}{\mathcal{J}} = \left[\begin{array}{c|cccc} & s{=}0 & s{=}1 & s{=}2 & \cdots \\ \hline t{=}0 & \mathcal{Y}_0 & \mathcal{Y}_1 & \mathcal{Y}_2 & \cdots \\ t{=}1 & \mathcal{E}_0{}'\mathcal{D}_0 & \mathcal{Y}_0 + \mathcal{E}_0{}'\mathcal{D}_1 & \mathcal{Y}_1 + \mathcal{E}_0{}'\mathcal{D}_2 & \cdots \\ t{=}2 & \mathcal{E}_1{}'\mathcal{D}_0 & \mathcal{E}_0{}'\mathcal{D}_0 + \mathcal{E}_1{}'\mathcal{D}_1 & \mathcal{Y}_0 + \mathcal{E}_0{}'\mathcal{D}_1 + \mathcal{E}_1{}'\mathcal{D}_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

**Example**: $\mathcal{J}_{2,2} = \underbrace{\mathcal{E}_1{}'\mathcal{D}_2}_{F_{2,2}} + \underbrace{\mathcal{E}_0{}'\mathcal{D}_1}_{F_{1,1}} + \underbrace{\mathcal{Y}_0}_{F_{0,0}}$ (sum along $\nwarrow$ diagonal)

**Cost**: $n_x T$ backward + $n_y(T-1)$ forward vs. direct: $n_x T^2 + n_x T^2$

Speed-up $\sim T \approx 300$

# Computing Times (Table 2 from the paper)

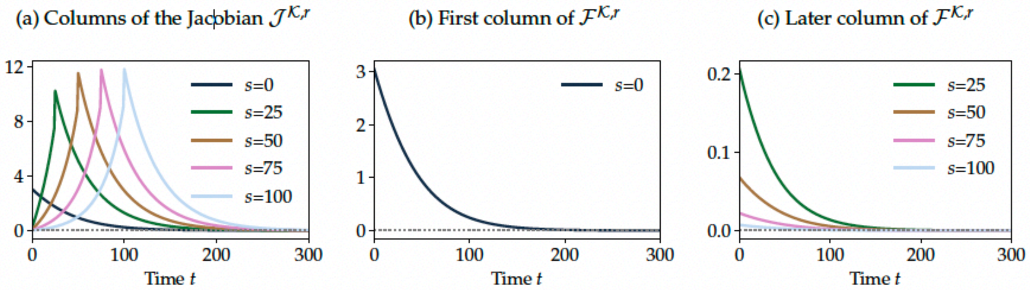
| | KS | HD KS | 1-asset HANK | 2-asset HANK |
|---|---|---|---|---|
| Direct | 21 s | 2,102 s | 156 s | 956 s |
| Fake news | 0.09 s | 10.5 s | 0.32 s | 3.50 s |
| $n_g$ | 3,500 | 250,000 | 3,500 | 10,500 |
| $n_x \times n_y$ Jacobians | 4 | 4 | 16 | 20 |

**Jacobians as sufficient statistics**

$\mathcal{J}$ captures all complexity from heterogeneity. Once computed, the underlying $n_g$-dimensional distribution no longer matters for aggregate dynamics

# Visualizing $\mathcal{J}^{K,r}$ and $\mathcal{F}^{K,r}$: Krusell–Smith

Figure 2: Jacobian $\mathcal{J}^{K,r}$ and fake news matrix $\mathcal{F}^{K,r}$ in the Krusell-Smith model.




Asymptotic time invariance: $\mathcal{J}_{t,s} \approx \mathcal{J}_{t-1,s-1}$ for large $t, s$ (columns are shifted versions of each other)

# General Equilibrium via DAGs

# The Problem: From PE Jacobians to GE Impulse Responses

We now have $\mathcal{J}^{\tilde{K},r}$ and $\mathcal{J}^{\tilde{K},w}$ from the fake news algorithm

But we need the GE impulse response matrix $G = -H_K^{-1} H_Z$

How do we get from one to the other?

**Brute force**: write the full system $\mathbf{F}(\mathbf{X}, \mathbf{Z}) = 0$ with $\mathbf{X} = (K, r, w, \tilde{K})$ – that's $n_x = 4$ sequences, so a $4T \times 4T$ system to differentiate and invert

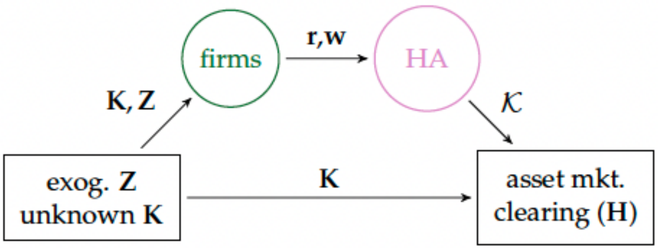
**Better**: notice that $r$ and $w$ are *determined* by $(K, Z)$ via the firm FOCs (5)–(6), and $\tilde{K}$ is determined by $(r, w)$ via the capital function $\mathcal{K}$ from eq. (4). So we can substitute out intermediate variables and reduce to:

$$H(\mathbf{K}, \mathbf{Z}) = 0 \qquad \text{– just } n_u = 1 \text{ unknown, a } T \times T \text{ system}$$

The **DAG** formalizes this substitution and automates the chain rule

# DAG: Krusell–Smith

Figure 3: DAG representation of Krusell-Smith economy




$n_u = 1$ unknown ($K$), 1 target ($H = \tilde{K} - K = 0$), 1 exogenous shock ($Z$)

Read the DAG left to right: each block takes inputs and produces outputs

- **Circles** = blocks that compute something (simple or HA)
- **Arrows** = flow of variables between blocks
- **Rectangles** = starting points (unknowns, shocks) and endpoints (targets)

# Following the DAG: From Unknowns to Targets

**Goal**: express the target $H$ as a function of $(K, Z)$ by tracing the arrows

**Step 1 — Firms block** (simple, analytical):

$$(K, Z) \xrightarrow{\text{eqs. (5)–(6)}} (r, w)$$

Jacobians: $\mathcal{J}^{r,K}$, $\mathcal{J}^{r,Z}$, $\mathcal{J}^{w,K}$, $\mathcal{J}^{w,Z}$ — analytical, from firm FOCs

**Step 2 — HA block** (fake news algorithm):

$$(r, w) \xrightarrow{\text{eqs. (10)–(12)}} \tilde{K}$$

Jacobians: $\mathcal{J}^{\tilde{K},r}$, $\mathcal{J}^{\tilde{K},w}$ — computed once via fake news

**Step 3 — Market clearing** (trivial):

$$(\tilde{K}, K) \rightarrow H = \tilde{K} - K = 0$$

# Composing Jacobians Along the DAG

Apply the chain rule following the arrows to build $H_K$ and $H_Z$:

**W.r.t. the unknown** $K$ (two paths: $K \to r \to \tilde{K} \to H$ and $K \to w \to \tilde{K} \to H$, plus direct):

$$H_K = \underbrace{\mathcal{J}^{H,\tilde{K}}}_{I} \cdot \Big( \underbrace{\mathcal{J}^{\tilde{K},r}}_{\text{fake news}} \cdot \underbrace{\mathcal{J}^{r,K}}_{\text{firms}} + \underbrace{\mathcal{J}^{\tilde{K},w}}_{\text{fake news}} \cdot \underbrace{\mathcal{J}^{w,K}}_{\text{firms}} \Big) + \underbrace{\mathcal{J}^{H,K}}_{-I}$$

**W.r.t. the exogenous shock** $Z$ (two paths: $Z \to r \to \tilde{K} \to H$ and $Z \to w \to \tilde{K} \to H$):

$$H_Z = \underbrace{\mathcal{J}^{H,\tilde{K}}}_{I} \cdot \Big( \underbrace{\mathcal{J}^{\tilde{K},r}}_{\text{fake news}} \cdot \underbrace{\mathcal{J}^{r,Z}}_{\text{firms}} + \underbrace{\mathcal{J}^{\tilde{K},w}}_{\text{fake news}} \cdot \underbrace{\mathcal{J}^{w,Z}}_{\text{firms}} \Big)$$

**Solve directly** — no guessing, no iteration:

$$d\mathbf{K} = -H_K^{-1} H_Z \, d\mathbf{Z}$$

Block Jacobians computed once. Composing and inverting: milliseconds.

# Scaling Up: From KS to HANK

The same logic applies to any model — just more blocks and more arrows

| | KS | 1-asset HANK | 2-asset HANK |
|---|---|---|---|
| Endogenous variables $n_x$ | 3 | 9 | 21 |
| Unknowns $n_u$ (after DAG) | 1 | 3 | 3 |
| Variables substituted out | 2 | 6 | 18 |
| GE computation time | 3.3 ms | 50.6 ms | 173.5 ms |

> The DAG is modular: to change the model, swap out a block (e.g. flexible $\to$ sticky prices), recompute only the affected Jacobians, and re-solve for GE

Any new shock path $d\mathbf{Z}$: just a matrix-vector multiply $G \cdot d\mathbf{Z}$

# Application

# One-Asset HANK

# One-Asset HANK: Model Overview

Textbook NK model + HA household sector (McKay, Nakamura, Steinsson 2016)

**Households**: relative to KS, also choose hours $n_{it}$. Budget constraint:

$$c_{it} + a_{it} = (1 + r_t)a_{it-1} + w_t e_{it} n_{it} - \tau_t \bar{\tau}(e_{it}) + d_t \bar{d}(e_{it})$$

$a_{it} \geq \underline{a}$. Taxes $\tau_t$ and dividends $d_t$ distributed according to incidence rules $\bar{\tau}(e)$, $\bar{d}(e)$

**Firms**: monopolistic competition, Rotemberg pricing. Phillips curve:

$$\log(1 + \pi_t) = \kappa \left( \frac{w_t}{F'(N_t)} - \frac{1}{\mu} \right) + \frac{1}{1 + r_{t+1}} \frac{Y_{t+1}}{Y_t} \log(1 + \pi_{t+1}) \tag{51}$$

Dividends: $d_t = Y_t - w_t N_t - \psi_t$ (output net of labor and price adjustment costs)

# One-Asset HANK: Policy and Market Clearing

**Fiscal policy**: balanced budget

$$\tau_t = r_t B + G_t$$

Government issues bonds $B$, spends $G_t$, adjusts lump-sum tax $\tau_t$

**Monetary policy**: Taylor rule + Fisher equation

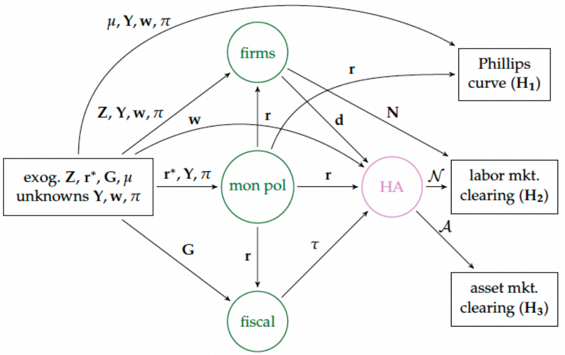
$$i_t = r_t^* + \phi\pi_t + \phi_y(Y_t - Y_{ss}), \qquad r_t = \frac{1 + i_{t-1}}{1 + \pi_t} - 1$$

**Market clearing**:

- Goods: $Y_t = \int c_{it}\, di + G_t + \psi_t$
- Assets: $B = \int a_{it}\, di$
- Labor: $N_t = \int e_{it} n_{it}\, di$

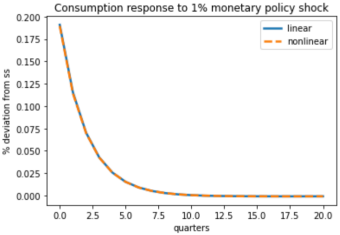
# One-Asset HANK: DAG

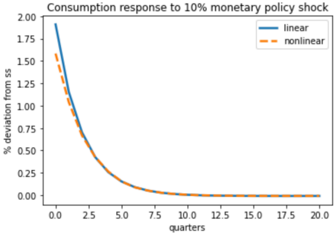
Figure C.1: DAG representation of one-asset HANK economy




- $n_u = 3$ unknowns: $(Y, w, \pi)$ $n_z = 4$ shocks: $(Z, r^*, G, \mu)$
- 3 targets: Phillips curve $(H_1)$, labor market $(H_2)$, asset market $(H_3)$
- 6 intermediate variables substituted out along the DAG
- Only the HA block requires the fake news algorithm

# One-Asset HANK: Consumption Response to Monetary Policy




- For small shocks (left): linear ≈ nonlinear — linearization is accurate
- For large shocks (right): differences appear on impact — higher-order terms matter
- SSJ also provides a good starting point for solving the nonlinear transition via Newton methods

# Summary: The SSJ + DAG recipe for solving HA models

1. **General HA representation**: backward–forward–aggregation, eqs. (10)–(12)
2. **Fake News Algorithm**: computes PE Jacobians $\mathcal{J}$ in $O(T)$
   - $\rightarrow$ Policy symmetry (Lemma 1) $\Rightarrow$ single backward pass
   - $\rightarrow$ $F_{t,s} = \mathcal{E}'_{t-1}\mathcal{D}_s$ (Lemma 2) $\Rightarrow$ outer product structure
   - $\rightarrow$ $\mathcal{J}_{t,s} = \sum_k F_{t-k,s-k}$ (Proposition 1) $\Rightarrow$ cumulative sum
3. **DAG**: decompose model into blocks, compose Jacobians via chain rule, reduce $n_x \rightarrow n_u$
4. **Solve**: $d\mathbf{U} = -H_U^{-1} H_Z \, d\mathbf{Z}$ – no iteration, milliseconds

Once we have the GE impulse responses ($G$ matrix):

- Simulation via MA representation (BKM insight from last lecture)
- Analytical second moments, likelihood $\Rightarrow$ Bayesian estimation
- New shock paths: just $G \cdot d\mathbf{Z}$ (microseconds)

# Readings for this lecture

**Required:**

- Auclert, Bardóczy, Rognlie, and Straub (2021), "Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models," *Econometrica*, 89(5), 2375–2408
  `doi:10.3982/ECTA17434`
  $\rightarrow$ This lecture: Sections 3–4, Appendix B.2

**Application background:**

- McKay, Nakamura, and Steinsson (2016), "The Power of Forward Guidance Revisited," *AER*, 106(10), 3133–3158
  `doi:10.1257/aer.20150063`
- Kaplan, Moll, and Violante (2018), "Monetary Policy According to HANK," *AER*, 108(3), 697–743
  `doi:10.1257/aer.20160042`

**Code:** `https://github.com/shade-econ/sequence-jacobian`

# Readings for the Final Exam

1. Krusell and Smith (1998), "Income and Wealth Heterogeneity in the Macroeconomy," *JPE*, 106(5), 867–896
   **`doi:10.1086/250034`**
2. Khan and Thomas (2008), "Idiosyncratic Shocks and the Role of Nonconvexities in Plant and Aggregate Investment Dynamics," *Econometrica*, 76(2), 395–436
   **`doi:10.1111/j.1468-0262.2008.00837.x`**
3. Khan and Thomas (2013), "Credit Shocks and Aggregate Fluctuations in an Economy with Production Heterogeneity," *JPE*, 121(6), 1055–1107
   **`doi:10.1086/674142`**
4. Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2018), "Really Uncertain Business Cycles," *Econometrica*, 86(3), 1031–1065
   **`doi:10.3982/ECTA10927`**
5. McKay, Nakamura, and Steinsson (2016), "The Power of Forward Guidance Revisited," *AER*, 106(10), 3133–3158
   **`doi:10.1257/aer.20150063`**
6. Arellano, Bai, and Kehoe (2019), "Financial Frictions and Fluctuations in Volatility," *JPE*, 127(5), 2049–2103
   **`doi:10.1086/701792`**
7. Auclert, Bardóczy, Rognlie, and Straub (2021), "Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models," *Econometrica*, 89(5), 2375–2408
   **`doi:10.3982/ECTA17434`**