Endogenous Firm Scope via Organizational Complexity

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Research Seminar

November 2, 2026

Scalable Capability, Bounded Scope

- Intangibles are a rising share of firms' capital, and they are non-rival. (Crouzet, Eberly, Eisfeldt & Papanikolaou 2022)
- Firm scope is tight and related: from the smallest to the largest asset quintile, assets rise about 500-fold, mean scope only 1.6-fold; firms expand into related markets. (Hoberg & Phillips 2025)
- Even Amazon (healthcare) and Google (hardware) stall in distant markets: why doesn't a handful of firms span the whole economy?

What bounds firm scope?

This Paper: Firm Scope under Organizational Complexity

- Organizational complexity: the cumulative distance of the markets a firm serves from its core. It sets the price of headquarters attention in every one of them.
- Two forces set the scope of the firm:
  - the return to the non-rival blueprint: one blueprint earns in every market it reaches
  - the complexity cost: how different the markets are, not only how many
- Expansion is non-separable: firms grow along related lines and stop where complexity bites.
- Implications:
  - missing giants: Pareto capability, but complexity cuts the far tail of scope twentyfold
  - competition: tighter complexity lowers the wage of attention; small firms expand

Contribution: A Market's Cost Depends on the Footprint

What bounds a firm's scope, and does a new market raise the cost of the others?

- Each market pays its own cost: **no**, markets are added one at a time.
  Melitz 2003; Eckel & Neary 2010; Mayer, Melitz & Ottaviano 2014; Arkolakis, Ganapati & Muendler 2021; Jiang & Rubinton 2026 (a total cost in the count)
- A shared input spread over the count: **yes**, by how many markets are added.
  Lucas 1978 (span); Garicano 2000 (time); Nocke & Yeaple 2014 (organizational capital)
- **This paper**, attention priced by complexity: **yes**, by which markets are added.
  Composition precedents: Crémer, Garicano & Prat 2007; Gumpert, Steimer & Antoni 2022

Intangibles are non-rival and scalable; what bounds their reach?

Crouzet, Eberly, Eisfeldt & Papanikolaou 2022, 2024; De Ridder 2024; Argente et al. 2025

– there, a cost per market or in the count bounds the reach

– **This paper**: attention, the rival input that deploys it, is priced by heterogeneity

▶ Closest papers

Outline

1. Environment: markets, organizational distance, complexity, the price of attention.
2. What bounds scope: the single firm's problem and the shape of its footprint.
3. Complexity and count: a parametrized economy, and how the two differ.
4. The missing giants: calibration to the scope distribution of U.S. firms.
5. Competition: many firms, a price field and a wage of attention, two results.

Environment

Environment · What bounds scope · Complexity and count · The missing giants · Competition

Markets

Market space. A measure space (S, Σ, μ): markets $s \in S$, measurable sets of markets $E \in \Sigma$, and their mass $\mu(E)$.

- μ is non-atomic: any E with $\mu(E) > 0$ contains E' of any smaller mass.
- Examples of a market s:
  - espresso machines for German cafés
  - cloud storage for hospitals
  - small-business loans in Ohio

▶ Assumptions

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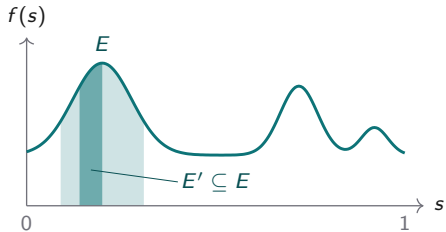


Illustration: $S = [0, 1]$ with an arbitrary density f (none is assumed); $\mu(E)$ is the whole shaded area, $\mu(E')$ its dark part.

▶ Assumptions

Market Characteristics and Capacity

Characteristics. Each market s has characteristics $\xi(s) \in \mathbb{R}^k$, such as geography, technology, customers, regulation; ξ is measurable. For Borel $B \subseteq \mathbb{R}^k$, $\nu(B) \equiv \mu\big(\xi^{-1}(B)\big)$.

- $\nu(B)$: mass of markets with $\xi(s) \in B$.

Capacity. Each market s has local resources $\bar{X}(s) > 0$, its size; $\bar{X}$ is measurable.

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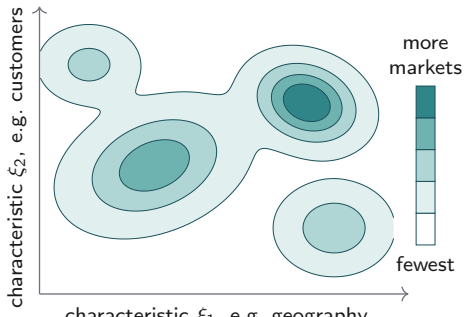


Illustration: $k = 2$. Shading: the density of ν, darker where more markets have those characteristics.

Heterogeneity: Distance from the Core

Organizational distance. The firm has a core $c \in S$. A Borel $\delta : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}_+$, $\delta(y, y) = 0$, not necessarily a metric, gives

$$d(s) = \delta\big(\xi(s), \xi(c)\big), \qquad d(c) = 0.$$

- δ is the organizational technology: which differences are costly to bridge.
- $d(s) > 0$ for μ-a.e. s: the core is a null point.

▶ Assumptions

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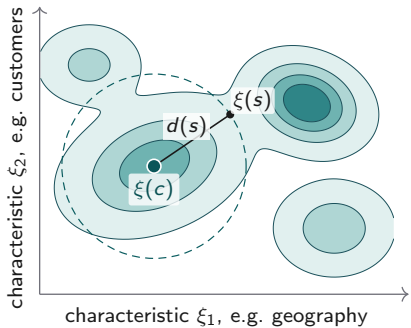


Illustration as before, δ Euclidean; dashed circle: markets at the same distance as s.

▶ Assumptions

Central Object: Organizational Complexity

Definition. For a footprint $M \in \Sigma$,

$$C(M) \equiv \int_M d(s)\,\mu(ds) \in [0,\infty] :$$

nonnegative, monotone, and additive over disjoint footprints.

- For Borel $B \subseteq \mathbb{R}^k$ and $M = \xi^{-1}(B)$:

$$\mu(M) = \nu(B), \quad C(M) = \int_B \delta\big(y, \xi(c)\big)\,\nu(dy).$$

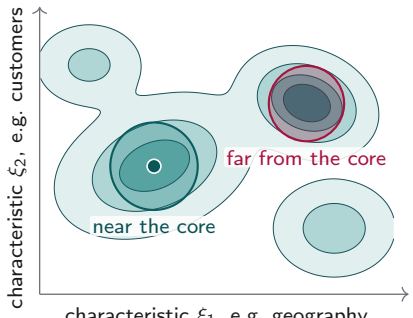
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nonnegative, monotone, and additive over disjoint footprints.

- Same mass of markets, and the far disc has the higher complexity.



Each disc is a footprint: the markets whose characteristics lie in it.

The Blueprint and Its Deployment

- Blueprint. Quality, the firm's capability, $z \geq 0$, created at headquarters with $\chi(\underset{+}{z})$ units of attention; non-rival: one z serves every market.

- Deployment. In market s, headquarters attention $h(s)$ and local effort $\ell(s)$ adapt the blueprint; its portability θ falls with distance:

$$z(s) \;=\; \theta\big(\underset{+}{a(s)},\, \underset{-}{d(s)}\big)\, z, \qquad a(s) \;=\; A\big(\underset{+}{h(s)},\, \underset{+}{\ell(s)}\big).$$

- Payoff. $R\big(\underset{+}{z(s)},\, \underset{+}{x(s)}\big)$ in market s, with local production input $x(s)$.

▶ Assumptions

The Central Cost: The Price of Headquarters Attention

- Attention bill. Creation and adaptation draw on one pool of headquarters attention, every unit priced at $q(C(M))$:

$$q\big(C(M)\big)\Big(\underbrace{\chi(z)}_{\text{creation}} + \underbrace{\int_M h(s)\,\mu(ds)}_{\text{adaptation}}\Big)$$

- Price of attention. $q : \mathbb{R}_+ \to \mathbb{R}_+$, $q(0) = 1$, $q' > 0$, $q(C) \to \infty$ as $C \to \infty$: its argument is complexity $C(M)$.

▶ Assumptions

The Firm's Problem

Given its core c, the firm chooses its capability z, footprint M, and allocations $\{x(s), \ell(s), h(s)\}$:

$$V(c) \equiv \max_{z,\, M,\, \{x,\ell,h\}} \int_M R\big(z(s), x(s)\big)\, \mu(ds) \;-\; q\big(C(M)\big)\Big(\chi(z) + \int_M h(s)\, \mu(ds)\Big)$$

$$\text{s.t.} \quad x(s) + \ell(s) \le \bar{X}(s), \quad x, \ell, h \ge 0 \quad \forall\, s \in M$$

$$\text{where} \quad z(s) = \theta\big(a(s), d(s)\big)\, z, \qquad d(s) = \delta\big(\xi(s), \xi(c)\big)$$

$$a(s) = A\big(h(s), \ell(s)\big)$$

$$M = \{\, s : x(s) + \ell(s) + h(s) > 0 \,\}$$

▶ Assumptions ▶ Existence, uniqueness

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Payoff: $R(z(s), x(s))$ in every served market.

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$$M = \{\, s : x(s) + \ell(s) + h(s) > 0 \,\}$$

Attention bill: every unit priced at $q(C(M))$, which rises with each market added.

▶ Assumptions ▶ Existence, uniqueness

What Bounds Scope

What Is Bounded Is Complexity

Fix capability z. What is bounded at the optimum?

Proposition (bounded complexity). If payoffs are integrable at fixed z, then at any (minimal-support) optimum

$$C\big(M^\star(z)\big) = \int_{M^\star(z)} d(s)\,\mu(ds) \;<\; \infty.$$

- The mass of markets served, $\mu(M^\star)$, may be infinite.
- For every $r > 0$: $\mu\big(\{s \in M^\star : d(s) \geq r\}\big) \leq C(M^\star)/r$.
- Example: $S = [0,1]$, $c = 0$, $d(s) = s$, $\mu(ds) = s^{-3/2}ds$: $\mu(S) = \infty$, $C(S) = 2$.

▶ Assumptions ▶ details ▶ Existence

Expansion Is Non-Separable

The value of market s at finite complexity C (blueprint cost aside):

$$\Pi_s(z;\, C) \equiv \sup_{x,\ell,h\geq 0,\ x+\ell\leq \bar{X}(s)} \big\{ R(z(s), x) - q(C)\, h \big\}.$$

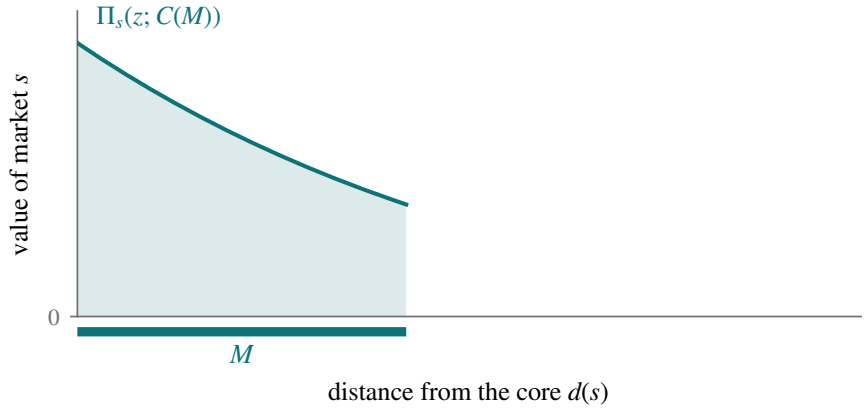
Proposition (complexity cost). $\Pi_s(z;\, C)$ is weakly decreasing in C; hence adding markets Δ weakly lowers every incumbent:

$$\Pi_s\big(z;\, C(M \cup \Delta)\big) \ \leq\ \Pi_s\big(z;\, C(M)\big) \quad \forall\, s \in M.$$

- A market's value depends on the whole footprint, through C: selection is non-separable.

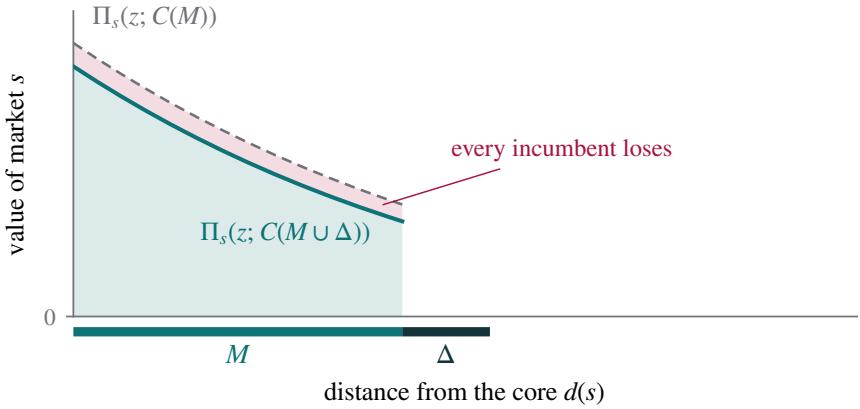
▶ Assumptions ▶ details

Adding Markets: Near versus Far



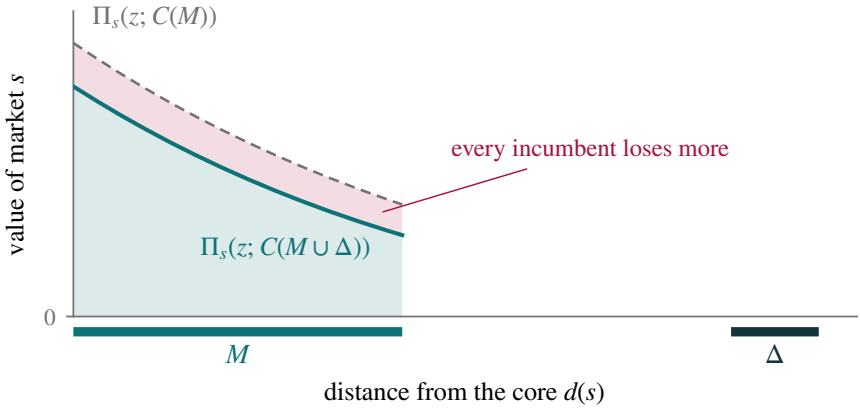
Footprint M, blueprint z given: a market's value falls with its distance from the core.

Adding Markets: Near versus Far



Add markets Δ next to M: C rises, attention costs more, and every incumbent loses.

Adding Markets: Near versus Far



Add the same mass far from the core: C rises more, and every incumbent loses more.

Related Expansion: A Cutoff in Heterogeneity

Which markets does the firm serve?

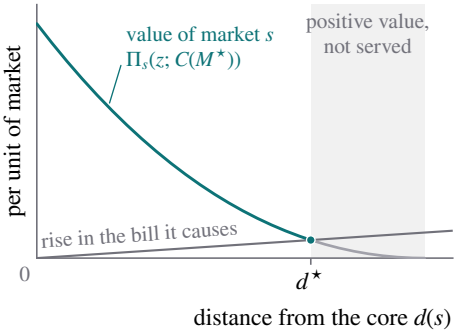
Proposition (cutoff selection). If a market's value $\Pi_s(z; C)$ is weakly decreasing in $d(s)$, there is a threshold $d^\star \in [0, \infty]$ with

$$\{\, s : d(s) < d^\star \,\} \;\subseteq\; M^\star(z) \;\subseteq\; \{\, s : d(s) \le d^\star \,\} \qquad (\mu\text{-a.e.}).$$

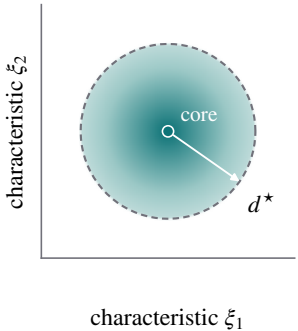
- The footprint is a ball around the core: related expansion (Hoberg & Phillips 2025).
- No fixed cost per market: the price of attention sets $d^\star$.

▶ details

The Cutoff and the Core–Periphery Allocation



A market is served while its value covers the rise in the attention bill it causes.



Inside the footprint, headquarters attention $h^\star(s)$ falls with distance from the core.

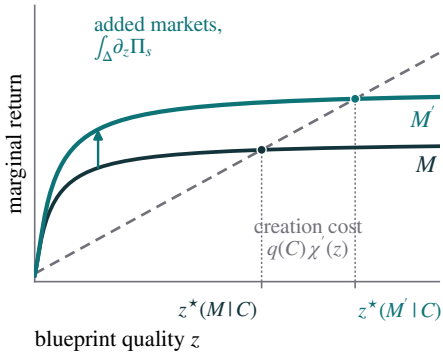
▶ details

Scale Raises the Return to the Blueprint

Let the firm choose z. Hold the price of attention $q(C)$ fixed and expand M to $M' = M \cup \Delta$.

Proposition (returns to blueprint scale). At fixed C, with $\Pi_s = \Pi_s(z; C)$,

$$\partial_z \int_{M'} \Pi_s \, d\mu = \partial_z \int_M \Pi_s \, d\mu + \underbrace{\int_\Delta \partial_z \Pi_s \, d\mu}_{\substack{\text{added markets} \\ \geq 0}}$$



Shapes illustrative. An interior $z^\star$ needs the marginal return above $q(C)\chi'(z)$ somewhere and below it eventually; the ordering of $z^\star$ needs only the upward shift.

▶ Assumptions ▶ details

Expansion: Deployment Gain and Complexity Loss

Conditional profit: footprint M fixed, attention priced at any q, everything else optimal:

$$\widehat{V}(M;q) \equiv \sup_{z,\{x,\ell,h\}} \left\{ \int_M R\big(z(s),x(s)\big)\,\mu(ds) - q\Big[\chi(z) + \int_M h(s)\,\mu(ds)\Big] \right\}.$$

The firm's value is $V(c) = \sup_M \widehat{V}\big(M; q(C(M))\big)$: each footprint at its own price.

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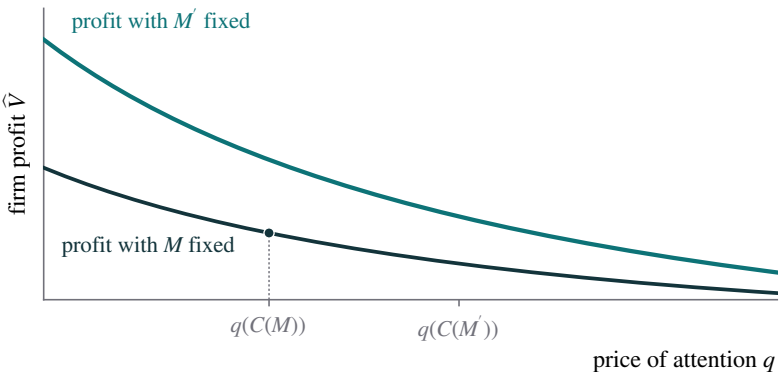
The firm's value is $V(c) = \sup_M \widehat{V}\big(M; q(C(M))\big)$: each footprint at its own price.

Proposition (expansion). $\widehat{V}(M;q)$ is weakly decreasing and convex in q; for $M' = M \cup \Delta$ and $q_0 = q(C(M)) \le q_1 = q(C(M'))$,

$$\widehat{V}(M';q_1) - \widehat{V}(M;q_0) = \underbrace{\widehat{V}(M';q_0) - \widehat{V}(M;q_0)}_{\text{deployment gain} \quad \ge 0} + \underbrace{\widehat{V}(M';q_1) - \widehat{V}(M';q_0)}_{\text{complexity loss} \quad \le 0}.$$

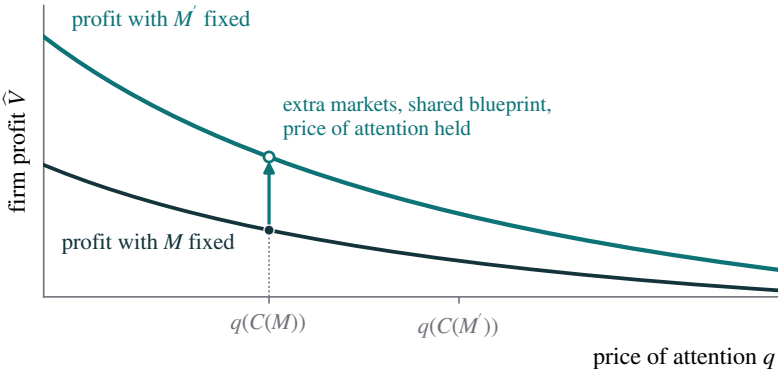
▶ Proof

Expansion: Shared Blueprint against Complexity



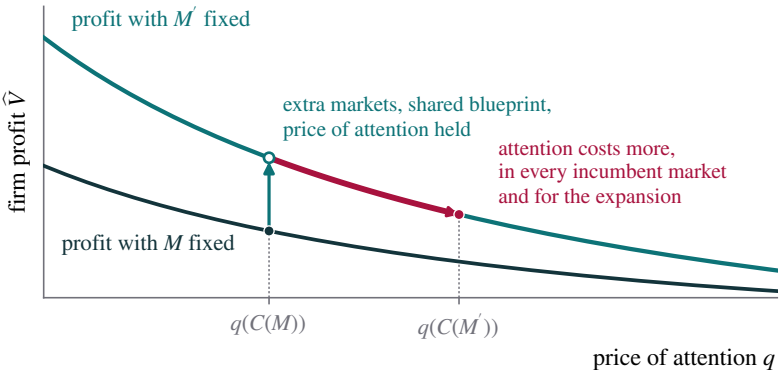
Each curve fixes its footprint and varies the price of attention, blueprint and adaptation re-optimized; a footprint's own price picks its point.

Expansion: Shared Blueprint against Complexity



Deployment gain: the larger footprint at the old price, $\widehat{V}(M'; q_0) - \widehat{V}(M; q_0) \geq 0$.

Expansion: Shared Blueprint against Complexity



Complexity loss: the same footprint at its own, higher price, $\widehat{V}(M'; q_1) - \widehat{V}(M'; q_0) \leq 0$.
Expansion pays while the gain exceeds the loss.

Complexity and Count

Economy P: A Parametrized Environment

- Markets. A line of characteristics, $S = \mathbb{R}$, $\xi(s) = s$: $\nu = \mu$ has a bounded density $f \geq 0$, which may vanish on whole intervals; core c; $d(s) = |s - c|$.
- Footprint M: count $N = \mu(M)$, complexity $C = C(M)$, mean distance $\bar{d} = C/N$. A ball of reach D, with $\rho_c(u) = f(c + u) + f(c - u)$ markets at distance u:

$$N(D; c) = \int_0^D \rho_c(u)\, du, \qquad C(D; c) = \int_0^D u\, \rho_c(u)\, du.$$

Example: uniform $f \equiv \frac{1}{2}$ gives $N = D$, $C = D^2/2$, $\bar{d} = D/2$.

▶ Assumptions ▶ details ▶ Existence, uniqueness

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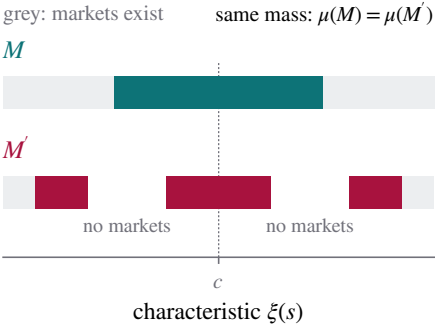
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  Example: uniform $f \equiv \frac{1}{2}$ gives $N = D$, $C = D^2/2$, $\bar{d} = D/2$.
- Payoff and capacity. $R(z(s), x) = z(s)\, x$, $\bar{X} \equiv 1$.
- Adaptation and portability. $A(h, \ell) = h$, $\theta(a, d) = \left(1 - e^{-a}\right) e^{-d/\lambda}$, $\lambda > 0$.
- Price of attention. $q(C) = \left(1 + \kappa C\right)^{\phi}$, $\kappa, \phi > 0$.
- Blueprint. z given, $\chi \equiv 0$.

▶ Assumptions ▶ details ▶ Existence, uniqueness

Separating Complexity from Count

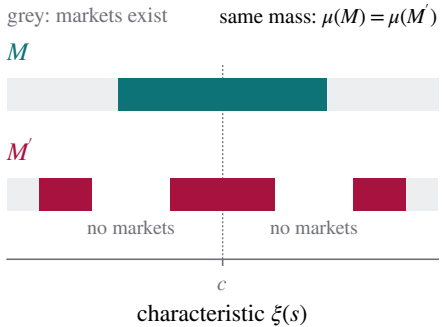
Two firms, same z, equal mass $\mu(M) = \mu(M')$; near the second core, markets run out.



▶ Formal statement

Separating Complexity from Count

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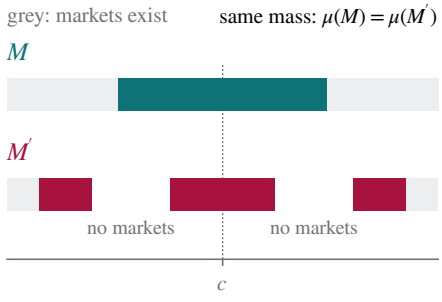


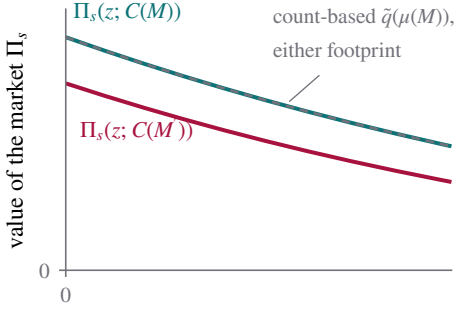
Left: each serves every market within reach; the second crosses the hole, so $C(M') > C(M)$.

▶ Formal statement

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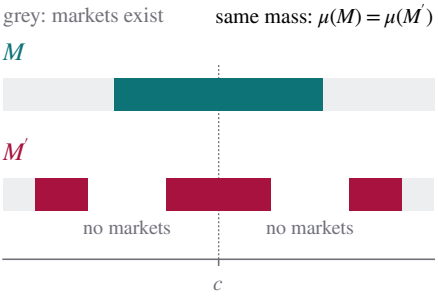


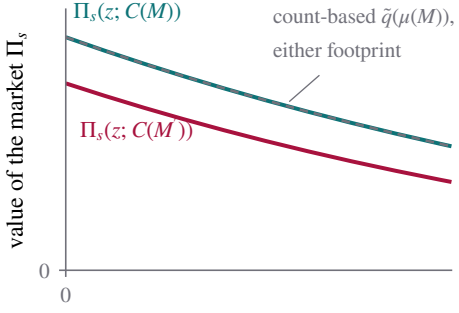
Right: $q(C(M')) > q(C(M))$, so every market in both footprints is worth less under M'. A count-based price of attention $\tilde{q}(\mu(M))$ gives both one price.

▶ Formal statement

Separating Complexity from Count

Two firms, same z, equal mass $\mu(M) = \mu(M')$; near the second core, markets run out.





Under a uniform density the count pins down complexity and the two cannot be told apart. They separate only where equal counts carry different complexity, as here.

▶ Formal statement

Complexity versus Count: What Tells Them Apart

Blueprint z and the incumbent's own distance fixed.

	A new market raises the incumbents' cost?	**An incumbent's value depends on**
A cost paid market by market Melitz; Eckel–Neary; Jiang–Rubinton	no	its own market
Count-based price of attention $\tilde{q}(N)$ Lucas; Garicano; Nocke–Yeaple	yes, by how many	N
Complexity $q(C)$, $C = N\bar{d}$	yes, by which	N and $\bar{d}$

▶ Formal statement

The Missing Giants

Environment · What bounds scope · Complexity and count · **The missing giants** · Competition

Many Firms

Firms. Firms i share the market space, the technology, and δ. Firm i has a core $c_i \in S$ and a capability z_i, both given:

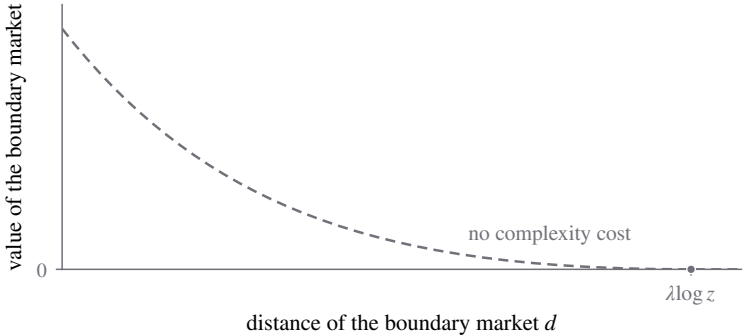
$$d_i(s) = \delta\big(\xi(s), \xi(c_i)\big), \qquad z_i \sim G.$$

- In this section firms do not interact: each solves the single-firm problem of economy P at its own core.
- Uniform benchmark: every core is alike, so scope is $D^\star(z_i)$ and the scope distribution is the image of G.
- G: Pareto with tail index α_z in the calibration.

▶ What the uniform benchmark identifies ▶ details

Complexity Stops the Firm Before Portability Does

Serve every market within distance d; follow the boundary market as d grows.

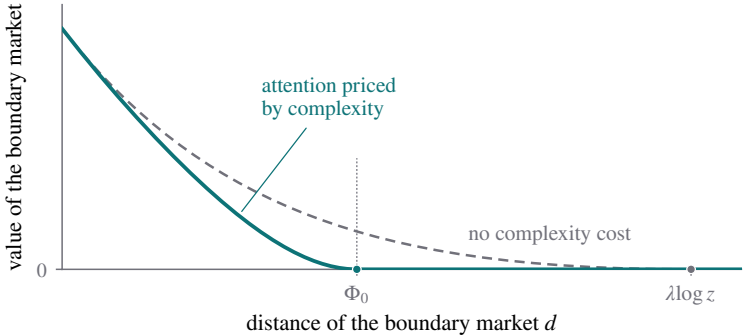


No complexity cost ($q \equiv 1$): the blueprint stops deploying at $\lambda \log z$, the limit set by portability alone.

▶ Assumptions ▶ details ▶ Existence, uniqueness

Complexity Stops the Firm Before Portability Does

Serve every market within distance d; follow the boundary market as d grows.

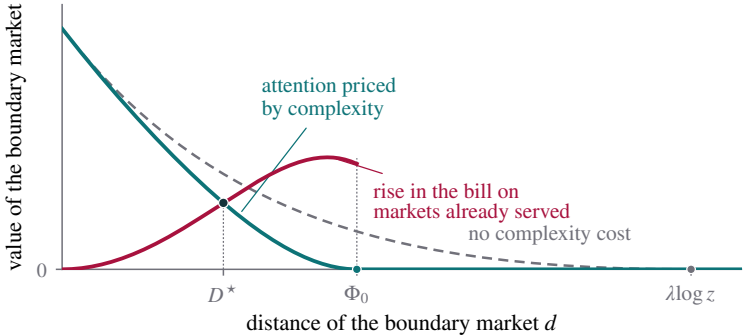


Frontier: where a firm that ignored non-separability would stop, each market judged on its own at the price its inclusion implies: $z\, e^{-\Phi_0/\lambda} = q\big(C(\Phi_0; c)\big)$.

▶ Assumptions ▶ details ▶ Existence, uniqueness

Complexity Stops the Firm Before Portability Does

Serve every market within distance d; follow the boundary market as d grows.



Fully internalized: the boundary market must also cover the rise in the bill on every market already served. For $z > 1$: $D^\star(z;c) < \Phi_0(z;c) < \lambda \log z$.

▶ Assumptions ▶ details ▶ Existence, uniqueness

Calibration to Hoberg–Phillips (2025)

Targeted moment	Data	Model
Mean scope	7.546	7.566
Median scope	6.000	6.124
Std. dev. of scope	5.800	5.653
Mean scope, asset quintile 1	5.870	5.809
Mean scope, asset quintile 2	6.890	6.595
Mean scope, asset quintile 3	7.610	7.241
Mean scope, asset quintile 4	8.100	8.058
Mean scope, asset quintile 5	9.260	10.127
Corr(scope, log assets)	0.291	0.279
Std. dev. of log assets	2.130	2.129

D2V-Scope: how many of 300 text-based industries a firm's 10-K is close to. Model: $D^\star$, one industry per unit of distance.

▶ parameters, full table ▶ identification ▶ fit ▶ in equilibrium

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The average firm serves 7.5 of 300 markets: scope is tightly bounded.

▶ parameters, full table ▶ identification ▶ fit ▶ in equilibrium

Calibration to Hoberg–Phillips (2025)

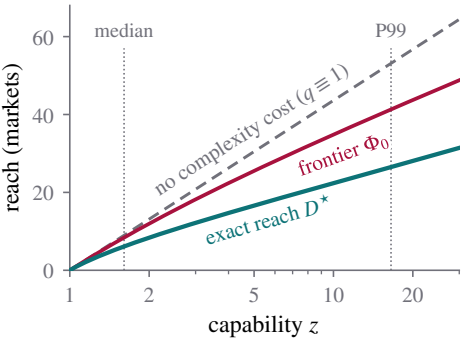
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Across asset quintiles, mean scope rises only from 5.9 to 9.3: size buys little scope.

▶ parameters, full table ▶ identification ▶ fit ▶ in equilibrium

Complexity Cuts the Tail Twentyfold

Same fitted Pareto capabilities, the three stopping rules:

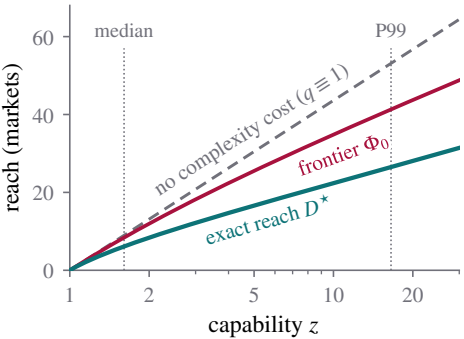


	$q \equiv 1$	Φ_0	$D^\star$
Mean scope	12.3	10.8	7.6
Std. dev. of scope	11.3	9.0	5.7
Scope at 10× capability	5.9×	4.9×	4.3×
Pr(scope $>$ 32)	0.064	0.033	0.0033

▶ Assumptions ▶ Tail compression ▶ Lucas, Melitz ▶ In equilibrium ▶ details

Complexity Cuts the Tail Twentyfold

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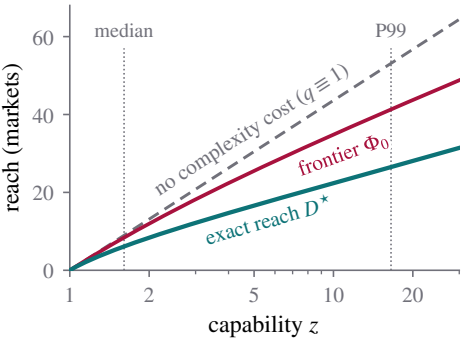
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Compression: against portability only ($q \equiv 1$), complexity cuts mean scope by 38% and halves its standard deviation.

▶ Assumptions ▶ Tail compression ▶ Lucas, Melitz ▶ In equilibrium ▶ details

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Mass above 32, the data's largest scope: twentyfold less than with portability only.

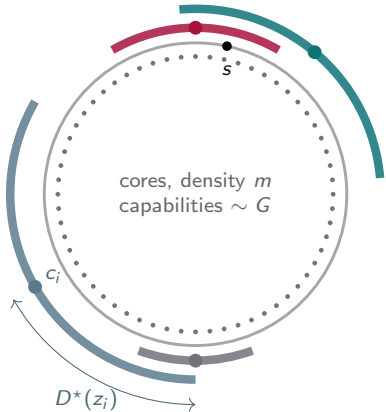
▶ Assumptions ▶ Tail compression ▶ Lucas, Melitz ▶ In equilibrium ▶ details

Competition and the Wage of Attention

Environment · What bounds scope · Complexity and count · The missing giants · **Competition**

Firms Meet on a Circle

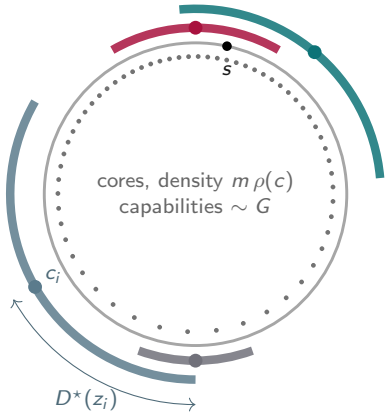
- The circle: bounded, no edge, so every market is alike; μ is half the arc length, so for any one firm it is economy P's line.
- A continuum of firms: cores over the circle at density $m\,\rho(c)$, capabilities iid from G. Spread evenly: $\rho \equiv 1$, the benchmark.
- Footprints overlap: a market is served by every firm whose arc covers it. Firms now meet through prices.



Dots: cores. Arcs: four footprints, each reaching $D^*(z_i)$ to either side of its core c_i; s is served by two.

Firms Meet on a Circle

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- A continuum of firms: cores over the circle at density $m\,\rho(c)$, capabilities iid from G. Spread unevenly: ρ varies, dense at the top, sparse at the bottom.
- Footprints overlap: a market is served by every firm whose arc covers it. Firms now meet through prices.



Dots: cores. Arcs: four footprints, each reaching $D^*(z_i)$ to either side of its core c_i; s is served by two.

The Firm's Problem under Competition

$$V(c_i, z_i) \equiv \max_{M_i,\, \{x_i, \ell_i, h_i\}} \int_{M_i} P(s)\, R\big(z_i(s), x_i(s)\big)\, \mu(ds) \;-\; w\, q\big(C(M_i)\big)\Big(\chi(z_i) + \int_{M_i} h_i(s)\, \mu(ds)\Big)$$

$$\text{s.t.} \quad x_i(s) + \ell_i(s) \le \bar{X}(s), \qquad x_i, \ell_i, h_i \ge 0 \quad \forall\, s \in M_i$$

$$\text{where} \quad z_i(s) = \theta\big(a_i(s), d_i(s)\big)\, z_i, \qquad d_i(s) = \delta\big(\xi(s), \xi(c_i)\big)$$

$$a_i(s) = A\big(h_i(s), \ell_i(s)\big), \qquad M_i = \{\, s : x_i(s) + \ell_i(s) + h_i(s) > 0 \,\}$$

- c_i and z_i given, as before; the technology is the same for every firm.
- $P(s)$, w: output R sells at a local price, attention is bought at a wage; both taken as given.
- Local inputs x_i, ℓ_i: chosen under the cap $\bar{X}(s)$, in fixed supply and unpriced, so that Part I carries over unchanged. Pricing them is the next step.

▶ Assumptions ▶ Existence ▶ Price taking

Closing the Model: A Static Household

$$\max_{T_0 \geq 0,\ Y(\cdot) \geq 0} \quad T_0 + \Omega \, \tfrac{\eta}{\eta - 1} \int_S Y(s)^{\frac{\eta - 1}{\eta}} \, \mu(ds)$$

$$\text{s.t.} \quad \int_S P(s)\, Y(s)\, \mu(ds) \;\leq\; w\,(\bar{T} - T_0) + \int V(c_i, z_i)\, di.$$

- One household owns the firms: time endowment $\bar{T}$, leisure T_0, the rest sold to firms as attention at the wage w; $\Omega > 0$ the weight on goods against leisure, $\eta > 1$ the elasticity of demand.
- Interior optimum: $P(s) = w\,\Omega\, Y(s)^{-1/\eta}$. Linear in leisure, so no income effects: profits move T_0, not demand.

Competitive Equilibrium: Exists, Unique, Efficient

Definition. A competitive equilibrium is a bounded price field $P(\cdot)$, a wage w, a plan (M_i, x_i, ℓ_i, h_i) for every firm i, and a household allocation $(T_0, Y(\cdot))$ such that
(i) every plan solves the firm's problem at $P(\cdot)$ and w;
(ii) $(T_0, Y(\cdot))$ solves the household's problem at $P(\cdot)$ and w;
(iii) goods clear in every market, and time clears:

$$Y(s) = \int R\big(z_i(s), x_i(s)\big)\, \mathbf{1}\{s \in M_i\}\, di,$$
$$\bar{T} - T_0 = \int q\big(C(M_i)\big)\big(\chi(z_i) + \textstyle\int_{M_i} h_i\, d\mu\big)\, di.$$

- $di = m\rho(c)\,\mu(dc)\,G(dz)$: the firm population; more cores near s, more sellers, lower $P(s)$.
- Cores spread evenly, $\rho \equiv 1$: a symmetric equilibrium, $P(\cdot) \equiv 1$, one wage w.
- Economy P, finite mean capability, $\rho \equiv 1$: exists, unique in prices and output, efficient.

▶ Assumptions ▶ Theorem ▶ Existence ▶ Uniqueness, efficiency

Result 1: The Ratio Rule

Facing any price field $P(\cdot)$, the firm serves

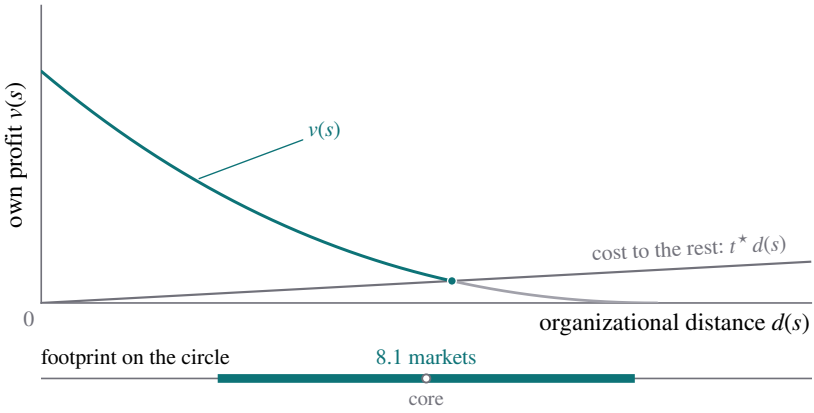
$$M^\star = \Big\{ s : \underbrace{v(s)}_{\text{own profit of } s} \geq \underbrace{t^\star\, d(s)}_{\substack{\text{rise in the attention bill} \\ \text{on every market already served}}} \Big\}.$$

$v(s) \equiv \Pi_s\big(z; C(M^\star)\big)$: the value of market s at price $P(s)$ and attention price $w\, q(C(M^\star))$;
$t^\star \equiv w\, q'\big(C(M^\star)\big) H^\star$: the marginal cost of complexity, $H^\star$ the firm's total attention.

▶ Assumptions ▶ Theorem

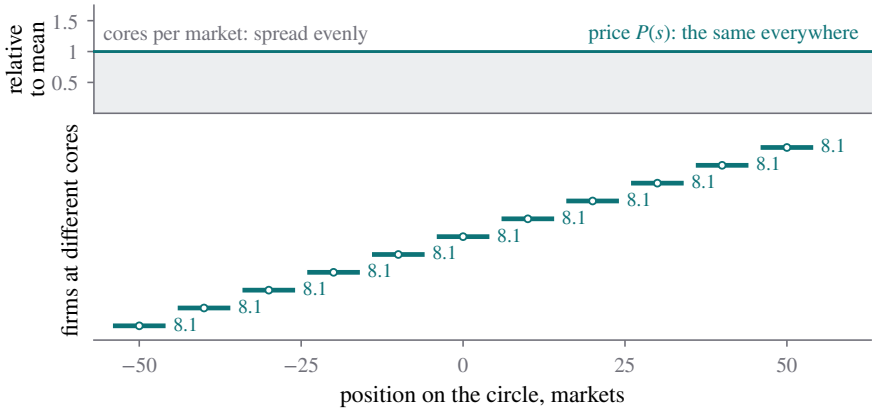
- Own profit must cover the cost to the rest: profitable but distant markets stay out.
- $t^\star$ rises with the attention behind it: larger firms are choosier per unit of distance.
- Prices enter own profit: footprints stretch toward high prices, where rivals are few.
- Part I's condition held at the ball's boundary; this one holds market by market.

Footprints Follow Prices



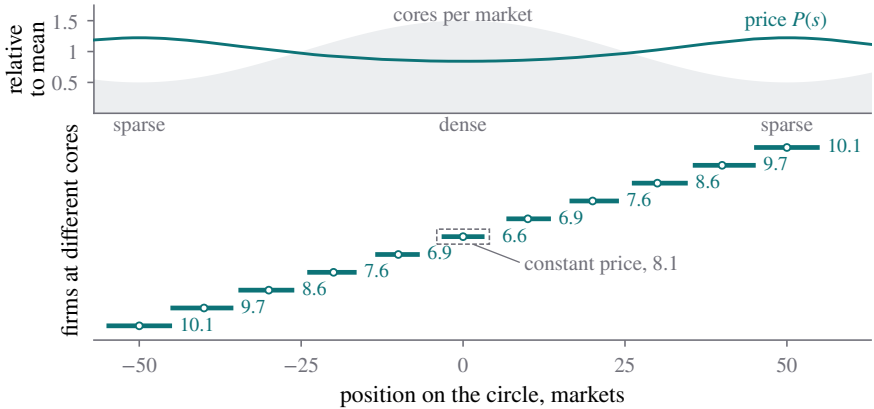
Constant price: one crossing, the footprint is an arc around the core.

Footprints Follow Prices



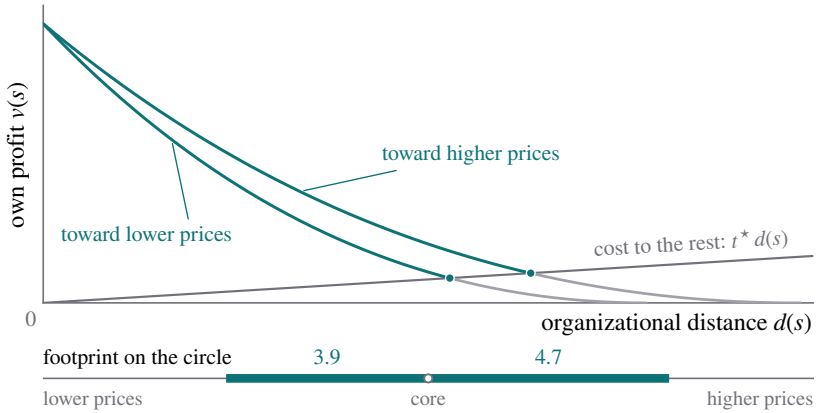
Cores spread evenly: the same price everywhere, the same arc everywhere.

Footprints Follow Prices



Cores spread unevenly: prices high where cores are scarce, footprints stretch there.

Footprints Follow Prices



Prices enter own profit only: the footprint leans toward the higher prices.

▶ Backup: a hole

Result 2: The Wage of Attention Damps Compression

Raise κ: the price of attention rises faster with complexity. Reach moves by

$$\frac{d \log D^\star}{d \log \kappa} = \underbrace{\varepsilon_{D,\kappa}}_{\substack{\text{fixed wage} \\ <0}} + \underbrace{\varepsilon_{D,w}\,\varepsilon_{w,\kappa}}_{\substack{\text{wage channel} \\ (-)(-)>0}} = \varepsilon_{D,w}\left[\varepsilon_{w,\kappa} + r_D(z)\right].$$

▶ Assumptions ▶ Theorem ▶ Damping ▶ Elasticities

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▶ Assumptions ▶ Theorem ▶ Damping ▶ Elasticities

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4. Exposure $r_D(z) \equiv \varepsilon_{D,\kappa}/\varepsilon_{D,w} > 0$, the firm's own: expands iff $r_D(z) < -\varepsilon_{w,\kappa}$.

▶ Assumptions ▶ Theorem ▶ Damping ▶ Elasticities

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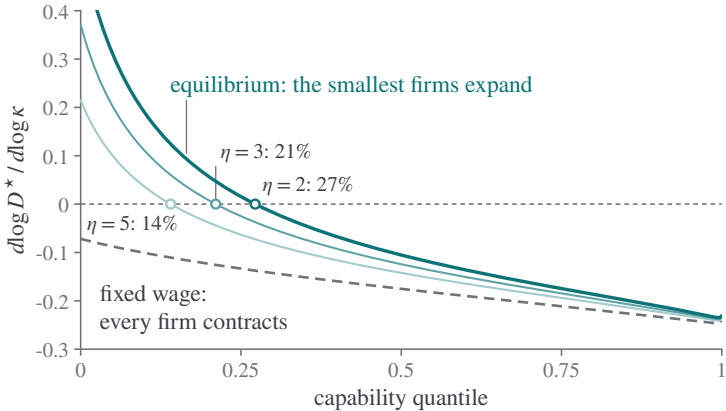
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The smallest firms expand (little complexity); the rest contract by less than at a fixed wage.

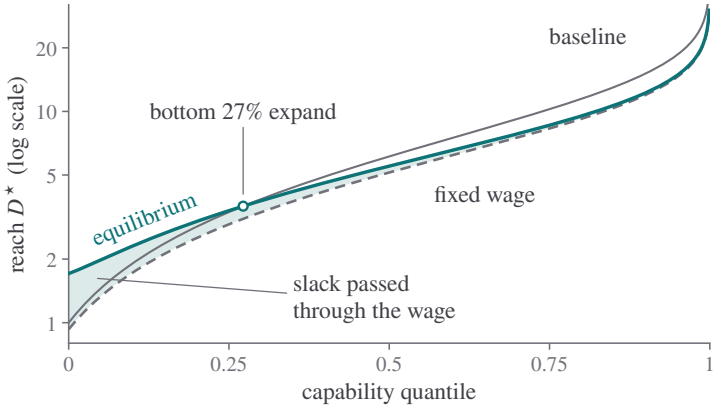
▶ Assumptions ▶ Theorem ▶ Damping ▶ Elasticities

The Lower Wage Lets the Smallest Firms Expand



Every firm fares better than at a fixed wage; the smallest firms expand.

The Lower Wage Lets the Smallest Firms Expand



Complexity binds on the large firms; the lower wage passes the slack to the small.

Key Takeaways

1. Neither a cost per market nor a cost in the count: scope is bounded by organizational complexity, the cumulative heterogeneity of the footprint.
2. Non-separable, related expansion: each added market raises the price of attention everywhere; firms grow along related lines and stop where complexity bites.
3. Missing giants: capability is Pareto, but complexity cuts the far tail of scope twentyfold.
4. Competition: the equilibrium is unique and efficient; footprints stretch toward high prices; the wage of attention damps compression and reverses it for the smallest firms.

▶ Concentration

Backup: Local and Aggregate Concentration (not presented)

Many Firms on One Market Space

Firms now meet: the space must be bounded, with no edge, so that every market is alike.

- The circle $\mathbb{T}_L$ of L markets, spread evenly; J firms, cores iid uniform, capabilities iid from G.
- Footprints overlap, yet firms do not interact: firm i serves its single-firm footprint, the arc of $D^\star(z_i)$ markets centered on c_i.

▶ One market space ◀ back

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Dots: cores. Arcs: footprints, on three tracks so overlaps show.

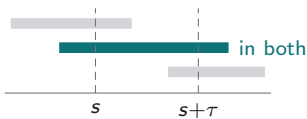
▶ One market space ◀ back

One Object for Both: The Overlap Kernel

Definition. Firm i's share of market s, $\omega_i(s) \geq 0$, sums to 1 over firms on served markets. The overlap kernel:

$$\mathcal{O}(\tau) = \frac{1}{L} \int_{\mathbb{T}_L} \sum_i \omega_i(s)\, \omega_i(s+\tau)\, ds.$$

- Pick a firm in a random market with probability equal to its share, then again in the market τ away: $\mathcal{O}(\tau) = \Pr(\text{same firm both times})$.
- $\tau = 0$: monopoly 1; n firms with equal shares (presence shares) $1/n$.



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Proposition (overlap-kernel identity). With $\bar{\omega}_i = \frac{1}{L}\int_{\mathbb{T}_L} \omega_i$ firm i's average share,

$$\mathcal{O}(0) = \overline{\mathrm{HHI}}_{\mathrm{loc}} \equiv \frac{1}{L} \int_{\mathbb{T}_L} \textstyle\sum_i \omega_i(s)^2\, ds, \qquad \frac{1}{L}\int_{\mathbb{T}_L} \mathcal{O}(\tau)\, d\tau = \mathrm{HHI}_{\mathrm{agg}} \equiv \textstyle\sum_i \bar{\omega}_i^2.$$

- Example: 100 firms, each alone on 1/100 of the circle: local 1, aggregate 1/100.

Compression Raises Local Concentration; Aggregate Barely Moves

Proposition (presence shares, many firms per market: $J \to \infty$).

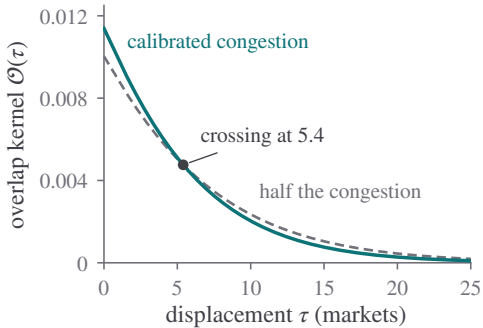
$$\frac{J\,\mathbb{E}[D^\star]}{L}\,\mathbb{E}\big[\overline{\mathrm{HHI}}_{\mathrm{loc}}\big] \to 1, \qquad J\,\mathbb{E}\big[\mathrm{HHI}_{\mathrm{agg}}\big] \to 1 + \mathrm{CV}^2,$$

$\mathbb{E}[D^\star]$: mean scope across firms; CV: its coefficient of variation.

- From half the calibrated κ to its full value, mean scope falls 8.58 $\to$ 7.57 and its CV 0.78 $\to$ 0.75.
- Calibration: $J = 3{,}519$ firms on $L = 300$ markets, about 89 per market.

▶ In equilibrium

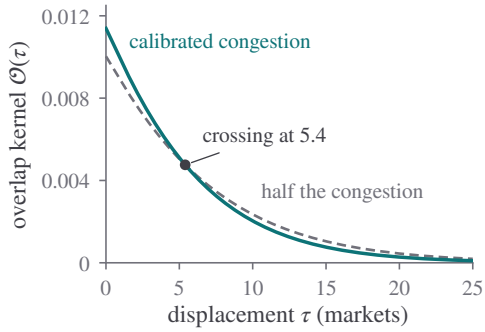
Compression Raises Local Concentration; Aggregate Barely Moves

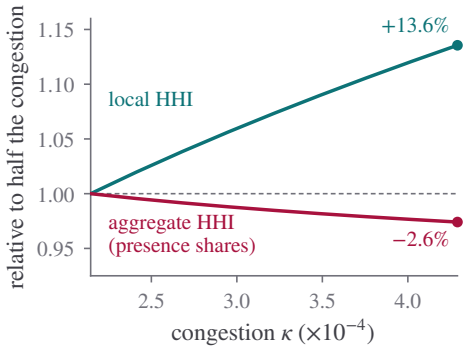


Left: overlap kernels, calibrated against half the κ: higher up to 5.4 markets, 0.7 of a mean footprint, lower beyond.

▶ In equilibrium

Compression Raises Local Concentration; Aggregate Barely Moves





Right: from half the calibrated κ to its full value, local HHI $+13.6\%$; aggregate HHI -2.6% with presence shares, $+0.9\%$ with deployment weights.

▶ In equilibrium

Appendix: Assumptions, Closed Forms, and Tables

Appendix: Standing Assumptions on the Primitives

◀ back

- **A1, markets.** (S, Σ, μ) σ-finite, complete, non-atomic; d and $\bar{X}$ measurable, finite, strictly positive μ-a.e.

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- **A4, integrable envelope.** For every finite z some $b_z \in L^1(\mu)$ bounds $\left[R\big(z\theta(A(h,\ell), d(s)), x\big)\right]_+$ for μ-a.e. s and all feasible (x,ℓ,h); $B(z) \equiv \int b_z \, d\mu$.

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- **A6, creation cost.** χ continuous, $\chi \geq 0$, $\chi(0) = 0$; when z is chosen, $B(z) - \chi(z) \to -\infty$.

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A1–A5 carry every fixed-z result, A6 is added when z is chosen; cutoff and core–periphery add the heterogeneity ordering (▶ here); Economy P specializes them (▶ here). No compact space, finite mass, concavity, or hole-free density is needed. Existence: ▶ here.

Appendix: Economy P and the Competitive Economy, Assumptions

◀ back

- **P1, markets.** The line, $\xi(s) = s$, with a bounded density $f \geq 0$ that may vanish on whole intervals; $d(s) = |s - c|$. Uniform means $f \equiv \frac{1}{2}$.

Appendix: Economy P and the Competitive Economy, Assumptions

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Appendix: Economy P and the Competitive Economy, Assumptions

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- **P3, blueprint.** Calibration: z given, $\chi \equiv 0$. When z is chosen: $\chi(z) = \chi_1 z^{\varepsilon_\chi}$, $\chi_1 > 0$, $\varepsilon_\chi > 1$, which gives A6. With $\chi \equiv 0$ and z free the value is unbounded, so the calibration keeps z given.

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- **E1, the competitive economy.** A compact metric market space with finite non-atomic μ (the circle, μ half the arc length); a continuum of firms, cores at density $m\rho(c)$, capabilities iid from G with finite mean; a bounded price field $P(\cdot)$ and a wage $w > 0$; household $\bar{T} > 0$, $\Omega > 0$, $\eta > 1$.

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The firm's problem under P1–P3, existence and uniqueness: ▶ here. The equilibrium under E1: ▶ here.

Appendix: The Firm's Problem Is Well Posed

◀ back

Theorem (existence). Under A1–A5, for every fixed z the firm's problem has a solution: a measurable plan with finite complexity and attention, one of them with an inclusion-minimal support; $-\chi(z) \leq V \leq B(z) - \chi(z)$. Under A1–A6 the problem with z chosen has a solution, with finite z.

Proof idea. Freeze the price of attention: each market's problem is a compact maximization (attention above $b_z(s)/r$ never pays at price r) with a measurable selection. The best footprint of complexity at most C is a threshold set in surplus per unit of distance, a bathtub problem on the cost measure $d\mu$ (non-atomic by A1), with value continuous in C. As $C \to \infty$ the price of attention diverges (A5) and every surplus vanishes (A2–A4), so the value over C attains its maximum; budget slack only lowers the bill, so the plan attains the true value. With z chosen, A6 confines z to a compact interval.

What each assumption rules out. Without continuity of R: a finite supremum, no maximizer; without A6's growth condition: value $+\infty$. **Uniqueness is not implied**: equidistant markets of positive mass give infinitely many optimal footprints under A1–A6; uniqueness comes from Economy P (▶ next).

Scope. Under competition (slide) returns are $P(s)R$ with P bounded and z given: the theorem applies.

Appendix: Economy P, Existence and Uniqueness

◀ back

Theorem (Economy P). Under P1–P2: (i) for any bounded density $f \geq 0$, holes allowed, and fixed z, a solution exists; it is empty for $z \leq 1$ and a ball of positive reach for $z > 1$. (ii) Uniform density, $\chi \equiv 0$: the solution is unique for every $\lambda, \kappa, \phi > 0$, with reach $D^\star(z) \in (0, \Phi_0(z))$ strictly increasing in z. (iii) Uniform density, z chosen, $\chi(z) = \chi_1 z^{\varepsilon_\chi}$, $\varepsilon_\chi > 1$: at most one operating global optimum, and one threshold on χ_1 separates profitable operation from inactivity (a tie at the threshold).

Proof idea. (i) A footprint of finite mass is dominated by the ball of the same mass: nearer markets are worth more at the old price and add less complexity; the ball's profit is continuous in its reach on $[0, \Phi_0]$. (ii) With $y = D/\lambda$, $Q(y) = q(\lambda^2 y^2/2)$, $a = yQ'/Q$, $t_z(y) = \log(z/Q(y)) - y$ and $g(t) = e^t - 1 - t$, the ball's profit has derivative $Q(y)\big[g(t_z(y)) - a(y)(t_z(y) + y/2)\big]$: the root of $g(t) = a(y)(t + y/2)$ rises in y while t_z falls, one crossing. (iii) Operating optima are interior and stationary; the stationary points with nonnegative profit form one curve along which the creation coefficient strictly decreases, so each χ_1 meets it at most once.

Limits. At fixed z with a creation requirement $k = \chi(z) > 0$: unique if $\phi \geq \frac{1}{2}$, a sharp bound. Not claimed for a non-uniform density. Unique means modulo null sets.

Appendix: Support, Zero Deployment, and Materially Distinct Markets

◀ back

Served markets. The footprint is the support of activity, $M = \{s : x(s) + \ell(s) + h(s) > 0\}$ up to null sets; among payoff-equivalent optima, a minimal support is selected: a market is included only when excluding it would lower the objective.

Assumption (No positive payoff without effective deployment). For any market s and feasible $x \le \bar{X}(s)$, $R(0, x) \le 0$: favorable local conditions alone generate no rent. Used only for the zero-blueprint case of the bounded-complexity proposition.

Corollary (finite mass of materially distinct markets). For any resolution $\tau > 0$ (an ex-post slicing, not a choice of the firm), with $M^\star_\tau \equiv \{s \in M^\star : d(s) \ge \tau\}$,

$$\mu(M^\star_\tau) \;\le\; \frac{1}{\tau} \int_{M^\star} d(s)\, \mu(ds) \;=\; \frac{C(M^\star)}{\tau} \;<\; \infty.$$

Count-based benchmark. With constant heterogeneity $d(s) \equiv \bar{d} > 0$, $C(M) = \bar{d}\, \mu(M)$: bounding complexity is bounding count, the span-of-control case.

Appendix: The Market-Level Value, and When the Cost Is Strict

◀ back

Definition (market-level value function). For a market s, blueprint z, and finite complexity C,

$$\Pi_s(z;C) \;\equiv\; \sup_{x,\ell,h\geq 0,\ x+\ell\leq \bar{X}(s)} \Big\{ R(z(s),x) - q(C)\,h \Big\}, \qquad z(s) = \theta\big(A(h,\ell), d(s)\big)\, z;$$

the creation cost $q(C)\chi(z)$ is common across markets and handled at the outer problem. Since q is increasing and $h \geq 0$, Π_s is weakly decreasing in C: this is all the non-separability proposition uses.

When is the complexity cost strict? The inequality is strict whenever $C(M \cup \Delta) > C(M)$, the inner problem attains its value, and every optimizer at $C(M)$ uses strictly positive headquarters attention. If an optimizer with $h = 0$ exists, a higher price of attention need not reduce the market's value.

In economy P the value of an active market is $q\,\psi(w)$ with $w = ze^{-d/\lambda}/q(C)$ and $\psi(w) = w - 1 - \log w$, and $\partial_q\big[q\,\psi(ze^{-d/\lambda}/q)\big] = -\log w < 0$ on the active set: the cost is strict in every active market.

Appendix: The Heterogeneity-Ordering Condition

◀ back

Assumption (Heterogeneity ordering). For any fixed (z, C), the market value $\Pi_s(z; C)$ is weakly decreasing in $d(s)$:

$$d(s_1) \leq d(s_2) \implies \Pi_{s_1}(z; C) \geq \Pi_{s_2}(z; C).$$

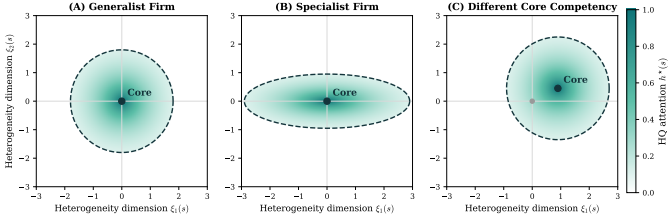
More heterogeneous markets are weakly less valuable to serve; this monotonicity is what turns selection into a threshold in d.

Sufficient primitive condition. Uniform local capacity $\bar{X}(s) = \bar{X}$: the inner feasible set is common across markets, effective deployment $\theta(A(h, \ell), d(s))\, z$ falls in d, so the objective $R(z(s), x) - q(C)\, h$ is pointwise larger at smaller d; taking suprema delivers the ordering.

Sufficient, not necessary: the ordering also holds for non-uniform $\bar{X}$ co-varying suitably with d. It restricts neither returns to scale nor within-market technology.

Proposition: Core–Periphery Allocation

◀ back



Proposition (core–periphery allocation). Under single-crossing of the conditional gross payoff $V(h, d; z)$ in headquarters attention and heterogeneity (decreasing differences), constant local capacity, and single-valued $h^\star(s)$: for any $s_1, s_2 \in M^\star$, $d(s_1) \leq d(s_2) \implies h^\star(s_1) \geq h^\star(s_2)$.

Attention $h^\star(s)$ over characteristic space; dashed curve: footprint boundary. Symmetric adaptation costs give a disc (a), costs accumulating slowly along ξ_1 an elongated footprint (b), a different core relocates it (c); attention declines with distance in every footprint.

Proposition: Returns to Blueprint Scale

◀ back

Proposition (returns to blueprint scale). Fix finite C. For any $z_2 \geq z_1 \geq 0$ and measurable $M_1 \subseteq M_2$,

$$\int_{M_2} \left[\Pi_s(z_2; C) - \Pi_s(z_1; C)\right] \mu(ds) \;\geq\; \int_{M_1} \left[\Pi_s(z_2; C) - \Pi_s(z_1; C)\right] \mu(ds);$$

under adaptation regularity and differentiation under the integral, also $\partial_z \int_{M_2} \Pi_s \, \mu(ds) \;\geq\; \partial_z \int_{M_1} \Pi_s \, \mu(ds)$.

Every market deploys the same blueprint, so the gain from improving it accumulates over the served set: scope and blueprint quality are complements at fixed complexity. The calibration sets $\chi_1 = 0$.

Where the firm stops. By the envelope theorem the blueprint adjustment has no first-order effect on firm value at the optimum: scope is set by the cutoff rule. The deployment gain's sign uses finite values and $R(0,0) = 0$; the identity needs neither.

Appendix: The Expansion Decomposition, Proof

◀ back

Decreasing and convex in q. For any one feasible plan on M, profit is affine in q with slope $-\left[\chi(z) + \int_M h\, d\mu\right] \leq 0$; $\widehat{V}(M; q)$ is the supremum of these affine functions over a feasible set that does not depend on q. Where differentiable, $\partial_q \widehat{V} = -\left[\chi(z^\star) + \int_M h^\star d\mu\right]$, minus the attention used.

The identity. Add and subtract $\widehat{V}(M'; q_0)$; exact whenever the three values are finite. The order of the two steps is a choice.

Complexity loss ≤ 0. $M \subseteq M'$ gives $C(M') \geq C(M)$, so $q_1 \geq q_0$, and $\widehat{V}(M'; \cdot)$ is weakly decreasing.

Deployment gain ≥ 0. Take any feasible plan on M and extend it to M': on Δ set $x = \ell = 0$ and $h_\epsilon(s) = \epsilon\, d(s) > 0$, so that M' is the support up to null sets. The added attention is $\epsilon\, C(\Delta)$; the added gross payoff $\int_\Delta R(z(s), 0)\, d\mu$ is ≥ 0 since R is weakly increasing and $R(0,0) = 0$. Hence $\widehat{V}(M'; q_0) \geq \widehat{V}(M; q_0) - q_0\, \epsilon\, C(\Delta)$ for every $\epsilon > 0$; let $\epsilon \to 0$. Uses $C(\Delta) < \infty$ and the supremum (no attainment is claimed).

Relation to the paper. The firm-level counterpart of the paper's incumbent decomposition, $\Delta_\Pi =$ blueprint adjustment $+$ the complexity term (Section 2.4). The deployment gain contains the new markets and the blueprint adjustment together.

Appendix: Economy P and the Market-Level Closed Forms

◀ back

Economy P. (P1) Markets on a characteristic line with density f, core c, $d(s) = |s - c|$; distance density $\rho_c(d) = f(c + d) + f(c - d)$; ball of reach D: $N = \int_0^D \rho_c$, $C = \int_0^D u\,\rho_c(u)\,du$, $\bar{d} = C/N$ (uniform: $N = D$, $C = D^2/2$). (P2) $\bar{X}(s) \equiv 1$. (P3) $R(z(s), x) = z(s)x$. (P4) $A(h, \ell) = h$, $\theta(a, d) = (1 - e^{-a})\, e^{-d/\lambda}$. (P5) $q(C) = (1 + \kappa C)^{\phi}$. (P6) $\chi(z) = \chi_1 z^{\varepsilon_\chi}$, with $\chi_1 = 0$ in the calibration.
Market-level problem. With $w(d) \equiv z\, e^{-d/\lambda}/q(C)$ and $\psi(w) \equiv w - 1 - \log w$: a market at distance d is active iff $w(d) > 1$, and then

$$h^\star(d) = \log w(d), \qquad z_d = q\,\big(w(d) - 1\big), \qquad \Pi_d(z; C) = q\,\psi\big(w(d)\big),$$

strictly decreasing in d and in C; every market-level object depends on the rest of the footprint only through $q(C)$.

Economy P satisfies every maintained assumption of Sections 1–2, so cutoff selection and the core–periphery allocation obtain; the general density f enters only through ρ_c, and the uniform benchmark is the case with closed forms and the case calibrated.

Appendix: Count versus Complexity, Formal Statements

◀ back

Proposition 3.2 (complexity is the sufficient statistic). In economy P, fix z. For every footprint of finite complexity and every market at distance d, the market-level outcomes (price of attention, attention, deployment, value) are functions of (z, d, C) alone: the rest of the footprint matters only through $C = N\bar{d}$.

Corollary 3.1 (count versus complexity). A technology is count-based if its q depends on the footprint only through N. If f is constant, economy P is observationally equivalent to a count-based technology on optimal cutoff footprints, with $\tilde{q}(N) \equiv q\big(N^2/(2\rho)\big)$, $\rho = 2f$ markets per unit of distance. If f is not constant, no count-based technology reproduces it: firms at cores with different local density and the same count have different complexity, hence different prices of attention, under economy P, and the same price under any count-based technology.

The illustration on the slide. Illustrative parameters, not the calibration. The first firm's density is uniform; the second's vanishes on an interval on each side of its core. Both footprints are cutoff sets, every available market within reach; the reaches are set by equal mass, not by optimality.

Appendix: The Exact Optimum

◀ back

Lemma (exact optimum). The frontier is $\Phi_0(z;c) = H_0^{-1}(z;c)$ with $H_0(d;c) = e^{d/\lambda}\big(1+\kappa\, C(d;c)\big)^{\phi}$, the capability required to serve a market at distance d. For every $z > 1$ an optimal footprint exists, every minimal-support optimum is a ball, and every optimal reach $D^\star \in (0, \Phi_0(z))$ at which ρ_c is continuous and positive satisfies

$$\Pi_{D^\star}(z;C) \;=\; q'(C)\, D^\star\Big[\chi(z) + \int_0^{D^\star} h^\star(s)\,\rho_c(s)\,ds\Big], \qquad C = C(D^\star),$$

equivalently $z\, e^{-D^\star/\lambda} = q(C)\, w^\star$ with a wedge $w^\star > 1$.

The boundary market must cover the rise in the attention bill on everything the firm already does; a count-based $\tilde{q}(N)$ has the same margin, $\Pi_{D^\star} = \tilde{q}'(N)[\chi + \int h^\star \rho_c]$, and on the matched path $\tilde{q}'(N) = q'(C)D^\star$, so the factor $D^\star$ alone does not separate them. The wedge, $0 < \Phi_0(z) - D^\star \le \lambda \log w^\star$, grows with capability: 1.14 at the median, 3.08 at P99. In the uniform benchmark with $\chi = 0$ the optimum is unique and $D^\star$ strictly increasing in z, for all $\lambda, \kappa, \phi > 0$.

Appendix: Parameters and Targeted Moments

Targeted moments	**HP**	**Model**
Mean scope	7.546	7.566
Median scope	6.000	6.124
Std. dev. of scope	5.800	5.653
Mean scope, asset quintile 1	5.870	5.809
Mean scope, asset quintile 2	6.890	6.595
Mean scope, asset quintile 3	7.610	7.241
Mean scope, asset quintile 4	8.100	8.058
Mean scope, asset quintile 5	9.260	10.127
Corr(scope, log assets)	0.291	0.279
Std. dev. of log assets	2.130	2.129
Untargeted checks		
Skewness of scope		1.58
Pr(scope > 32), data max 32		3.3×10^{-3}
Median sample max, $n = 101{,}535$		65
Composite slope $\gamma = \phi\kappa$		8.6×10^{-4}

Set, not calibrated	
Markets per unit of distance ρ_c	1
Local capacity $\bar{X}$	1
Capability lower bound $\underline{z}$	1
Blueprint creation χ_1	0
Complexity elasticity ϕ (on the profile)	2
Calibrated to the ten moments	
Pareto tail index α_z	1.678
Portability decay λ, industries	18.94
Complexity weight κ	4.29×10^{-4}
Size-noise dispersion σ	2.044

Moments: Hoberg–Phillips (2025, Tables 1, 2, 5). Scope is $D^\star(z)$ for firms with at least one industry ($z \geq z_1 = 1.063$); size $\log S = \log z + \sigma\varepsilon$. ϕ is fixed on the profile: the data pin $\gamma = \phi\kappa$. ◀ back

Appendix: What the Scope Distribution Identifies

◀ back

- (α_z, σ) are pinned jointly by the two size moments: $\mathrm{sd}(\log S) = \sqrt{1/\alpha_z^2 + \sigma^2}$ exactly under the size equation, and the scope–size correlation loads on the same pair; the profiled objective over α_z is several times its minimum at $\alpha_z = 1.2$ or 2.3.
- Without the size moments, (α_z, λ) would sit on a ridge: the frontier survival exponent $\alpha_z[m/\lambda + \phi \log(1 + \kappa m^2/2)]$ depends on four parameters through three composites. Calibrating the exact optimum helps, since the wedge depends on λ separately.
- (κ, ϕ) is the weak direction: the profiled objective falls by less than 1% as ϕ rises from 2 to 12, with κ adjusting so that $\gamma = \phi\kappa \approx 8 \times 10^{-4}$ is stable. The data pin the count-cost schedule $q(C(N)) = (1 + \kappa N^2/2)^{\phi}$, in effect $e^{\gamma C}$, not the pair behind it: the count-versus-complexity corollary operating inside the calibration. ϕ is fixed at the smallest value within 1% of the profile minimum.
- What would separate them is composition: conditional on capability, count, and own distance, incumbent outcomes vary with the mean distance of the rest of the footprint under economy P and under no count-based technology. Hoberg–Phillips tabulate operating-industry pairs by relatedness bin.

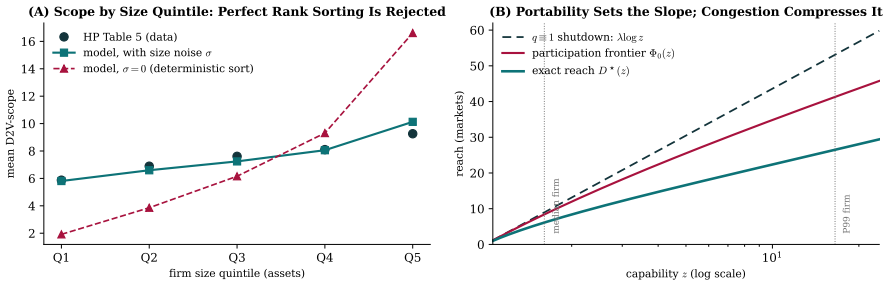
Appendix: What the Uniform Benchmark Can and Cannot Tell

◀ back

- Under a uniform density optimal footprints are balls, $C = N^2/2$: on optimal choices, complexity and a count-based price of attention of the induced shape, $\tilde{q}(N) = q(N^2/2)$, are observationally equivalent.
- So Part IV measures the curvature of the price of attention: the calibration pins $\gamma = \phi\kappa$, and ϕ anywhere from 2 to 12 fits within one percent (identification).
- What complexity adds in Part IV is the shape: a ball of reach D has complexity $D^2/2$, so the price rises with the square of reach. Distance-weighting delivers it; a count theory has to assume it.
- What complexity adds beyond Part IV is composition: conditional on capability, count, and own distance, an incumbent's outcome varies with the mean distance of the rest of the footprint under complexity and under no count-based technology (slide 21's hole, slide 32's lean).
- The test: Hoberg–Phillips tabulate operating-industry pairs by relatedness bin; composition conditional on count is the next step, not this calibration. A limitation of the calibration, not of the theory; a count-based schedule with free curvature would fit these nine moments too.

Appendix: Calibration Fit and the Compression of the Scope Mapping

◀ back



(a) Hoberg–Phillips Table 5: mean scope by asset quintile 5.87, 6.89, 7.61, 8.10, 9.26, between-quintile sd 1.14 against a total of 5.80. A deterministic size–capability relation sorts firms exactly by scope and pins a profile from 1.9 to 16.6, which the data reject; under $\log S = \log z + \sigma\varepsilon$ the quintile means are exact closed-form integrals ($\sigma = 2.04$). (b) Exact reach, frontier, and the portability-only mapping $\lambda \log z$ at the fitted parameters, median and 99th-percentile capabilities marked. One untargeted moment fails: the model places 3.3×10^{-3} above the observed maximum of 32, partly mechanical, D2V tops out at 32; flagged, not resolved.

Appendix: The Anatomy of Compression, Full Table

◀ back

	Portability only, $q \equiv 1$	Frontier, Φ_0	Full model, $D^\star$
Mean scope	12.28	10.81	7.57
Median scope	8.82	8.28	6.12
Std. dev. of scope	11.28	8.96	5.65
Scope ratio at tenfold capability	5.86	4.87	4.29
Pr(scope $>$ 32)	0.064	0.033	0.0033

All columns at the fitted (α_z, λ) and, where applicable, (κ, ϕ), conditional on at least one market. The ratio row is $D(10z)/D(z)$ at the full model's conditional median capability $z = 1.61$; the ninety-ninth-percentile capability is 16.5, 10.3 times the median. Portability decay sets the logarithmic slope; complexity works in levels, dispersion, and the tail. Within the complexity channel, the frontier accounts for about a quarter of the level compression in logs and internalization for three quarters: the wedge $w^\star$ is 1.14 at the median firm and 3.08 at the ninety-ninth percentile, and complexity is 4.7 and 16.7 percent of $\log H_0$ at the corresponding reaches.

Appendix: The Calibration in General Equilibrium

The levels are equilibrium outcomes. In the symmetric equilibrium every firm solves economy P at capability z/w. The moments pin neither Ω nor $\bar{T}$: set Ω so that $w = 1$ (demand $(w\Omega)^{\eta}$ equals supply at $w = 1$) and $\bar{T}$ above the firms' attention use. The calibrated cross-section is the equilibrium cross-section, firm by firm.

- The decomposition is at a fixed wage. Removing the complexity cost raises supply; in equilibrium the wage rises and pushes every firm back: the no-complexity-cost scope is below 12.3, the compression below 38% and less than twentyfold.
- How much the wage takes back. Marginal change in κ, the damping theorem: 29, 21, 14 percent at $\eta = 2, 3, 5$. Full removal of the complexity cost: a fixed point in w per η, not yet computed.
- The frontier Φ_0 is a rule no firm follows: no equilibrium counterpart; an equilibrium anatomy has two columns.

◀ back

Appendix: Tail Compression, Formal Statement

◀ back

Proposition 3.1 (tail compression). In economy P, for any measurable selection $D^\star$ from the optimal radii, if $\Pr(Z > x) = x^{-\alpha_z}$ for $x \geq 1$, then for every $m \geq 0$

$$\Pr\big(D^\star(Z;c) > m\big) \;\leq\; \Pr\big(\Phi_0(Z;c) > m\big) \;=\; H_0(m;c)^{-\alpha_z} \;\leq\; e^{-\alpha_z m/\lambda},$$

$$\Pr\big(N(D^\star;c) > m\big) \;\leq\; e^{-\alpha_z m/(\rho_{\max}\lambda)}\,:$$

reach and scope have exponentially bounded tails and every moment is finite, while capability is Pareto. Here $H_0(m;c) = e^{m/\lambda}\big(1+\kappa\, C(m;c)\big)^{\phi}$ and $\Phi_0 = H_0^{-1}$.

Proof. $D^\star \leq \Phi_0$ by the exact-optimum lemma; $\{\Phi_0(Z;c) > m\} = \{Z > H_0(m;c)\}$ since H_0 is strictly increasing with $H_0 \geq 1$; $H_0(m;c) \geq e^{m/\lambda}$ since $C \geq 0$; and $N(D;c) \leq \rho_{\max} D$. □

In the uniform benchmark the frontier tail is exact, $\Pr(\Phi_0(Z) > m) = e^{-\alpha_z m/\lambda}(1+\kappa m^2/2)^{-\alpha_z\phi}$; the exact reach is asymptotically a fixed fraction of the frontier and grows at the same logarithmic rate.

Appendix: Capability and Size in the Canonical Models

◀ back

- Span of control (Lucas 1978). A manager of talent z hires n workers at wage w and produces $z\,n^{\beta}$, $0<\beta<1$:
$$\beta z\, n^{\beta-1} = w \implies n(z) = \big(\beta z/w\big)^{1/(1-\beta)}.$$
Pareto talent, $\Pr(Z>x)=x^{-\alpha_z}$, gives $\Pr\big(n(Z)>m\big) = (\beta/w)^{\alpha_z}\, m^{-\alpha_z(1-\beta)}$: size is a power of talent with elasticity $1/(1-\beta)>1$, and the Pareto tail is passed on and thickened.

- Monopolistic competition (Melitz 2003). CES demand with elasticity $\sigma>1$ gives revenue $r(\varphi)\propto\varphi^{\sigma-1}$; Pareto productivity with index k gives $\Pr\big(r(\varphi)>m\big)\propto m^{-k/(\sigma-1)}$: again a power, again a Pareto tail.

- Economy P. The boundary market's return decays exponentially in distance and is priced at $q(C)\geq 1$:
$$z\,e^{-d/\lambda} > q(C) \geq 1 \implies D^{\star} < \Phi_0 \leq \lambda \log z, \qquad \Pr\big(D^{\star}(Z)>m\big) \leq e^{-\alpha_z m/\lambda}.$$

In each case the optimum equates the return of the last unit to its price: a return decaying polynomially ($n^{\beta-1}$, $\varphi^{\sigma-1}$) makes size a power of capability, one decaying exponentially in organizational distance ($e^{-d/\lambda}$) makes reach logarithmic, and a Pareto tail is compressed rather than passed on.

Appendix: Many Firms, One Market Space

◀ back

Many firms. Firm i has its own core $c_i \in S$; the market space, the characteristics and δ are common:

$$d_i(s) = \delta\big(\xi(s), \xi(c_i)\big).$$

- For the multi-firm results: $S = \mathbb{T}_L$, the circle of circumference L, with uniform μ; markets are identified with their attributes, so $d_i(s) = \delta(s, c_i)$.

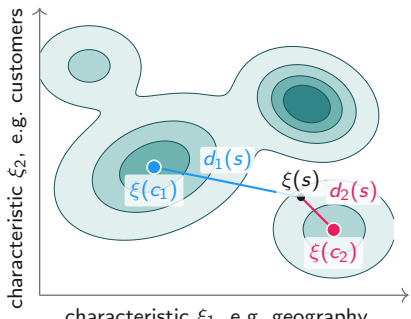


Illustration as on the environment slides; $d_2(s) = 0.17$, $d_1(s) = 0.55$.

Appendix: Closest Papers

◀ back

- **Gumpert, Steimer & Antoni (2022)**: CEO time spread over given sites; travel is costly.
  Here: the footprint is chosen; one complexity index prices all attention.
- **Crémer, Garicano & Prat (2007)**: broader scope, a coarser common language in every unit.
  Here: complexity over a continuum of markets, with scope, concentration, equilibrium.
- **Nocke & Yeaple (2014)**: a fixed stock of organizational capital spread over more products.
  Here: the cost depends on composition: where the firm expands matters at equal count.
- **Eckel & Neary (2010); Mayer, Melitz & Ottaviano (2014); Arkolakis, Ganapati & Muendler (2021)**: cost rises in own distance.
  Here: a market's cost depends on the whole footprint; markets are not chosen one at a time.
- **Jiang & Rubinton (2026)**: non-rival software; a total management cost in the count of establishments.
  Here: a new market raises the price of attention in every market, by its distance.
- **Azevedo, Weyl & White (2013)**: a continuum of agents convexifies indivisible goods.
  Here: the indivisible choice is the footprint; prices are fields over markets.

Appendix: Existence, Uniqueness, Efficiency

◀ back

At wage w the economy-P firm is economy P at effective capability z/w.

Theorem. Let $\mathbb{E}_G[z] < \infty$.
(i) Existence: with cores spread evenly, a symmetric equilibrium exists; its wage solves $Y(w) = (w\Omega)^{\eta}$.
(ii) Uniqueness: any two equilibria, symmetric or not, have the same prices and output at every market.
(iii) Efficiency: every equilibrium allocation solves the planner's problem.

- (ii): revealed preference against downward-sloping demand; no convexity, no asymmetric equilibrium.
- (iii): the complexity cost is internal and firms take prices as given; the wage channel is not a distortion.

▶ Existence ▶ Uniqueness, efficiency

Appendix: The Damping Theorem

▶ Firm-level elasticities ▶ Damping ◀ back

Theorem (damping). In economy P with cores spread evenly, a rise in κ lowers the wage of attention, with $\varepsilon_{w,\kappa} \in \left(-\frac{1}{2\eta}, 0\right)$, and moves the reach of every active firm of capability z by $d \log D^\star / d \log \kappa = \varepsilon_{D,w}\left[\varepsilon_{w,\kappa} + r_D(z)\right]$, $r_D(z) \equiv \varepsilon_{D,\kappa}/\varepsilon_{D,w} > 0$. (i) $r_D(z) \to 0$ as $z \downarrow w$: the smallest active firms expand. (ii) $r_D(z) > \phi\, s(z)$, $s = \kappa C^\star/(1+\kappa C^\star)$: every firm with $\phi\, s \geq 1/(2\eta)$ contracts. (iii) For $\phi \geq 1.034$, r_D is strictly increasing in z/w: one cutoff capability.

The formula. Reach depends on the wage only through z/w, so $D^\star = D^\star(z/w(\kappa); \kappa)$; differentiate and factor out $\varepsilon_{D,w}$. Both reach elasticities are negative: the fixed-wage term contracts the firm, the wage term expands it, and the firm expands iff $r_D(z) < -\varepsilon_{w,\kappa}$.

The wage bound. Market clearing, $(w\Omega)^\eta = Y(w, \kappa)$ with Y supply at output price one, gives $\varepsilon_{w,\kappa} = \varepsilon_{Y,\kappa}/(\eta - \varepsilon_{Y,w})$; supply falls in the wage and a rise in κ cuts each firm's output by less than half per log κ, so $-\varepsilon_{w,\kappa} < 1/(2\eta)$.

The parts. (i) Near participation reach is small and the attention bill barely congested, so r_D vanishes. (ii) The closed form of r_D at the optimum. (iii) An exact rational certificate; it fails at $\phi = 1.03$.

Appendix: The Wage Channel Damps Compression

◀ back

Raise κ. Supply falls, the wage of attention falls, every z/w rises.

Theorem (damping). $\bar{D}$: mean reach; $\varepsilon_{X,v}$: elasticity of X to v at fixed w or κ.

$$\varepsilon_{w,\kappa} \equiv \frac{d\log w}{d\log \kappa} = \frac{\varepsilon_{Y,\kappa}}{\eta - \varepsilon_{Y,w}} \in \left(-\frac{1}{2\eta}, 0\right), \qquad \frac{d\log \bar{D}}{d\log \kappa} = \varepsilon_{\bar{D},\kappa} \cdot \delta, \qquad \delta = 1 + \frac{\varepsilon_{\bar{D},w}}{\varepsilon_{\bar{D},\kappa}} \varepsilon_{w,\kappa} < 1.$$

calibration	fixed wage	$\eta = 2$	$\eta = 3$	$\eta = 5$
wage response $\varepsilon_{w,\kappa}$	0	−0.040	−0.029	−0.019
damping δ	1	0.71	0.79	0.86
mean reach, $d\log \bar{D}/d\log \kappa$	−0.202	−0.144	−0.160	−0.175

The wage channel removes 29, 21, 14 percent of the fixed-wage response.

▶ Firm-level elasticities ▶ Sign of δ

Appendix: Concentration in Equilibrium

◀ back

Many firms per market, presence weights, equilibrium reach $D = D^\star(z/w)$:

$$\text{local} \;\propto\; \frac{1}{\mathbb{E}[D]}, \qquad\qquad \text{aggregate} \;\propto\; 1 + \mathrm{CV}(D)^2.$$

response to $\log \kappa$	fixed wage	$\eta = 2$	$\eta = 3$	$\eta = 5$
local concentration	$+0.202$	$+0.144$	$+0.160$	$+0.175$
aggregate concentration	-0.039	-0.089	-0.075	-0.062
gap, log(local/aggregate)	$+0.241$	$+0.233$	$+0.235$	$+0.237$

- Local dominance rises less: the fixed-wage response times δ.
- Aggregate concentration falls more; the gap between them is nearly unchanged.

Appendix: The Ratio Rule

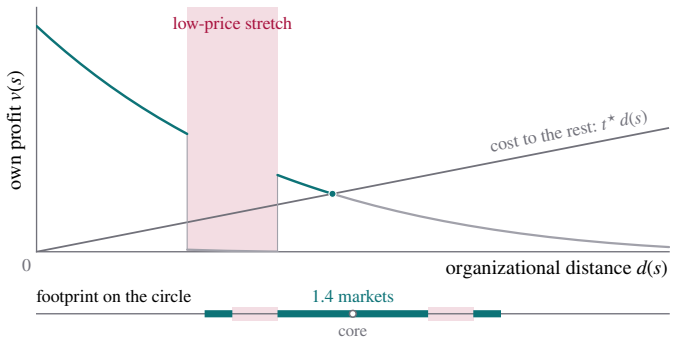
◀ back

Theorem (ratio rule). On a compact market space with finite μ, facing any bounded price field $P(\cdot)$, an optimal plan exists, and every optimal footprint is, up to markets at the threshold, $M^\star = \{s : \ v(s) \geq t^\star d(s)\}$, where $v(s) \equiv \Pi_s\big(z; C(M^\star)\big)$ at price $P(s)$ and price of attention $w\, q\big(C(M^\star)\big)$, and $t^\star \equiv w\, q'\big(C(M^\star)\big) H^\star$, $H^\star$ the firm's total attention.

Proof sketch (any price field, general environment). Fix the complexity C and the price of attention $q = q(C)$; the best footprint of complexity C maximizes $\int_M v(s;q)\,\mu(ds)$ subject to $\int_M d(s)\,\mu(ds) \leq C$, a bathtub problem whose solution is the superlevel set $\{v(\cdot;q) \geq t\, d(\cdot)\}$ filled to C (exchange argument; μ nonatomic). The firm's value is $\max_C \big[B(C; v(\cdot; q(C))) - q(C)\chi(z)\big]$ with B the bathtub value, continuous in C on $[0, C_{\max}]$, so an optimum exists (local attainment by A3, A4: well-posedness); every optimal plan is a bathtub set at its own complexity and attention price, and at an interior optimum $q'(C^\star)\big(\chi + \int_{M^\star} h^\star\big)$ lies between the ratio's supremum outside and infimum inside the footprint (add or remove a sliver of complexity ϵ). Direction: in economy P, $v(s;q)/d(s) = q\,\psi^+\big(P(s) z e^{-d(s)/\lambda}/q\big)/d(s)$ rises with $P(s)$ and falls with $d(s)$.

Backup: Prices Can Punch a Hole (not presented)

◀ back



A constructed price field, not an equilibrium: high everywhere except one stretch of distances, in illustrative units. There own profit falls below the ray, the cost to the rest; markets beyond the stretch stay above it, so the firm skips the stretch and serves two arcs. Under a price field a footprint need not be connected.

Appendix: Uniqueness, Efficiency, and the Open Step

◀ back

Uniqueness (bounded price fields; wage normalized to one). For two equilibria with price fields P_1, P_2 and optimal plans y_θ^1, y_θ^2: optimality of each plan at its own prices against the other gives $\int (P_1 - P_2)(y_\theta^1 - y_\theta^2)\, d\mu \geq 0$ per firm; integrating over firms (output bounded by $\bar{R}z$, $\mathbb{E}[z] < \infty$, Fubini), $\int_S (P_1 - P_2)(Y_1 - Y_2)\, d\mu \geq 0$. Demand $P = \Omega Y^{-1/\eta}$ is strictly decreasing market by market, so the same integral is ≤ 0 with equality only where $Y_1 = Y_2$. Hence $Y_1 = Y_2$ and $P_1 = P_2$ a.e.; total hours and profit then agree. No convexity of the technology enters.

Efficiency. Integrate firm optimality over firms and add the household's supergradient inequality $V(\tilde{Y}) \leq V(Y) + \int P(\tilde{Y} - Y)$: $V(\tilde{Y}) - \tilde{\mathbf{H}} \leq V(Y) - \mathbf{H}$ for every feasible assignment.

Open. Existence with cores spread unevenly: a planner optimum attained by pure plans on a saturated firm space is the equilibrium (first-order condition over the convex aggregate set, then a full switch of any positive mass of non-optimizing firms); attainment of the planner's maximum by pure plans is the open step, since a single firm's feasible set is not weakly closed (complexity is charged on the support).

Appendix: Price Taking, and a CES within Markets

◀ back

Why price taking. The question of this section is what a market for the organizational input does to scope. Price taking isolates that channel: the wage of attention is the only equilibrium feedback, and there is no strategic interaction to carry alongside it. Consistent with the environment, not imposed on conduct: firms and markets are nonatomic, every market has a continuum of sellers with homogeneous output, so $Y(s)$ and the time market do not depend on any one firm.

Where it is an approximation. At the calibration a market has about 89 sellers and capability is Pareto with tail index 1.68, infinite variance: the largest seller holds on average 17 percent of a market's output (median 14, 90th percentile 31). The concentration backup measures this; the competition section assumes it away.

A CES within each market. Differentiated sellers, within-market elasticity $\sigma > 1$: seller i earns $P(s)\, Y(s)^{1/\sigma}\, y_i(s)^{(\sigma-1)/\sigma}$ in market s; every seller stays negligible and charges the constant markup $\sigma/(\sigma-1)$ over its marginal attention cost. The firm's problem keeps Part I's form with payoff $[z_i(s)\, x]^{(\sigma-1)/\sigma}$ and effective capability $z_i\, [P(s) Y(s)^{1/\sigma} / w]^{\sigma/(\sigma-1)}$: prices enter through one number per market, so the ratio rule is unchanged; economy P's closed forms are not.

What it does not deliver. Constant markups, so market power independent of local concentration; that needs finitely many sellers per market, the next step. A uniform markup is a wedge on the attention margin, so efficiency and uniqueness need re-proof.

Appendix: The Symmetric Equilibrium, Existence

◀ back

Reduction. At output price one and wage w a firm's objective is $\int_M z\theta x - w\int_M q(C)h$: w times economy P's objective at capability $\zeta = z/w$ on the same plan space. Hence participation at $\zeta = 1$, a ball footprint, a unique reach $D^\star(\zeta)$ increasing in ζ, and attention $h^\star(u) = \log\big(\zeta e^{-u/\lambda}/q^\star\big)$ with $q^\star = q(D^{\star 2}/2)$.

Supply. Physical output of a firm, $Q(\zeta) = \int_0^{D^\star} (\zeta e^{-u/\lambda} - q^\star)\, du$ at $\zeta = z/w$; aggregate supply per market $Y(w) = m\int Q(z/w)\, dG(z)$.

- *Nonincreasing in w*, by revealed preference: for $\zeta_2 > \zeta_1$ with optimal plans and attention bills c_1, c_2, $\zeta_2 Q_2 - c_2 \geq \zeta_2 Q_1 - c_1$ and $\zeta_1 Q_1 - c_1 \geq \zeta_1 Q_2 - c_2$, so $(\zeta_2 - \zeta_1)(Q_2 - Q_1) \geq 0$.
- *Bounded and continuous*: output never exceeds λz, so $Y \leq m\lambda\, \mathbb{E}_G[z]$, and dominated convergence applies; $Y(w) \to 0$ as $w \to \infty$ because only capabilities above w are active.

Fixed point. Demand at output price one is $(w\Omega)^\eta$, strictly increasing from 0 to ∞, so $(w\Omega)^\eta - Y(w)$ crosses zero exactly once. Strict monotonicity and a firm-level supply elasticity above one follow from the first-order condition (next slide). Uniqueness is within symmetric equilibria; asymmetric price fields are open. The packets state the same results with attention as the numeraire, $\bar{P} = 1/w$.

Appendix: Firm-Level Elasticities from the First-Order Condition

◀ back

At the exact optimum, with $C^\star = D^{\star 2}/2$, write $\omega^\star = \zeta e^{-D^\star/\lambda}/q(C^\star)$ for the boundary market's ratio (the wedge $w^\star$ of the exact-optimum backup; w here is the wage), and

$$L = \log \omega^\star, \qquad \rho = \frac{D^\star}{2\lambda}, \qquad s = \frac{\kappa C^\star}{1+\kappa C^\star} \in (0,1), \qquad \gamma = 2\phi s, \qquad \psi = \omega^\star - 1 - L.$$

First-order condition: $\psi = \gamma\,(L+\rho)$: the boundary market's value equals the rise of the attention bill on everything behind it. **Three signs** (every $\lambda, \kappa, \phi > 0$):

$$b = (\omega^\star - 1) - \gamma > 0, \qquad A = \tfrac{\gamma}{2}\, b + (1-s)\,\psi > 0, \qquad T = \gamma b + 2(1-s)\psi + \rho\,(2b+\gamma) > 0.$$

b: effective capability raises the marginal value of reach; A: complexity lowers it; T: the optimum is regular. **Elasticities of reach** ($\zeta = z/w$, so the wage elasticity is minus the ζ elasticity):

$$\varepsilon_{D,w} = -\frac{b}{T} \in \Big(-\frac{\lambda}{D^\star}, 0\Big), \qquad \varepsilon_{D,\kappa} = -\frac{A}{T} \in \Big(-\frac{1}{2}, 0\Big), \qquad r_D = \frac{A}{b} = \frac{\gamma}{2} + (1-s)\frac{\psi}{b} > \phi s.$$

Monotonicity (M): r_D is strictly increasing in ζ for every $\phi \geq 1.034$ (exact rational certificate; fails at $\phi = 1.03$); (M) implies that $|\varepsilon_{D,w}|$ is strictly decreasing. A firm's output falls by less than half per $\log \kappa$, which gives $|\varepsilon_{w,\kappa}| < 1/(2\eta)$.

Appendix: When Mean Reach Falls

◀ back

Characterization. $\delta > 0$ iff $-\varepsilon_{w,\kappa} < \bar{r}_D$, the average of r_D over active firms weighted by $\zeta\, \partial_\zeta D^\star$: the wage decline against the firms' average exposure.

Sufficient. Identical capabilities; or every active firm contracts ($r_D > -\varepsilon_{w,\kappa}$ for G-almost every active firm), for which $\phi\, s(z_0/w) \geq 1/(2\eta)$ at the lowest capability z_0 suffices ($s = \kappa C^\star/(1+\kappa C^\star)$ as on the previous slide).

Can fail. $\lambda = \kappa = \phi = m = 1$, $\eta = 2$, mass 0.999 at $z = 1.001$ and 0.001 at $z = 10$, normalized to $w = 1$: $\varepsilon_{w,\kappa} = -0.069$, $\varepsilon_{\bar{D},w} = -513$, $\varepsilon_{\bar{D},\kappa} = -0.15$, $\delta = -241$: the productive type contracts and lowers the wage; the mass of barely active firms expands. The sign survives smoothing.

Pareto laws, calibrated technology, $\eta = 2$ (δ; the share of expanding firms in parentheses):

lowest capability z_1/w	1.005	1.063 (calibration)	2.0
$\alpha = 1.2$	0.86	0.87	0.94
$\alpha = 1.678$ (calibration)	0.67	0.71 (27%)	0.88 (0%)
$\alpha = 2.5$	0.54	0.62	0.85

Expansion shares run from 14 to 42 percent near the threshold and vanish once the lowest capability exceeds 1.3; a positive-damping theorem for the Pareto class is open.