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Online Appendix

"Optimal Fiscal Policy in a Model with Uninsurable Idiosyncratic Shocks"

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1.1 Detailed proof for uncertainty economy

It is useful to define $\tau_R^k \equiv r\tau^k/(1+r)$, and rewrite the following definition with τ_R^k instead of τ^k:

Definition 1 *A **tax distorted competitive equilibrium** is a vector* $\left(K, n_L, n_H, r, w; \tau^n, \tau_R^k, T\right)$ *such that*

1. (K, n_L, n_H) *solves*

$$\max_{a,n_L,n_H} u(\omega - a, \bar{n}) + \beta E\left[u(c_i, n_i)\right] \quad \text{s.t. } c_i = (1-\tau^n) w e_i n_i + \left(1-\tau_R^k\right)(1+r) a + T;$$

2. $r = f_K(K, N)$, $w = f_N(K, N)$, *where* $N = \pi e_L n_L + (1-\pi) e_H n_H$*;*

3. *and,* $\tau^n w N + \tau_R^k (1+r) K = G + T$.

Six equations determine such an equilibrium. The first order conditions of the agent's problem,

$$u_c(\omega - a, \bar{n}) = \beta\left(1-\tau_R^k\right)(1+r)\left[\pi u_c(c_L, n_L) + (1-\pi) u_c(c_H, n_H)\right], \tag{1.1}$$

$$u_n(c_L, n_L) = -(1-\tau^n) w e_L u_c(c_L, n_L), \tag{1.2}$$

$$u_n(c_H, n_H) = -(1-\tau^n) w e_H u_c(c_H, n_H); \tag{1.3}$$

the first order conditions of the firm's problem,

$$r = f_K(K, N), \tag{1.4}$$

$$w = f_N(K, N); \tag{1.5}$$

and the government's budget constraint

$$\tau^n w N + \tau_R^k (1+r) K = G + T \tag{1.6}$$

The results below make use of the following assumption:

Assumption 1 *No income effects on labor supply and constant Frisch elasticity,* κ*, i.e.*

$$u_{cn} - u_{cc}\frac{u_n}{u_c} = 0, \quad \text{and} \quad \frac{u_{cc}u_n}{n\left(u_{cc}u_{nn} - u_{cn}^2\right)} = \kappa.$$

Using equations (1.4) and (1.5) to substitute out for r and w we are left with a system of four equations that any vector $\left(K, n_L, n_H, \tau^n, \tau_R^k, T\right)$ of equilibrium values must satisfy. The two degrees of freedom are a result of the fact that the planner has three instruments $\left(\tau^n, \tau_R^k, T\right)$ that are restricted by one equation, the government's budget constraint. Defining welfare by

$$W \equiv u\left(\omega - K, \bar{n}\right) + \beta E\left[u\left(c_i, n_i\right)\right]$$

and totally differentiating the four equilibrium equations together with this definition and making the appropriate simplifications using Assumption 1 we obtain the following equation[1]:

$$dW = \Theta^n d\tau^n + \Theta^k d\tau_R^k,$$

where Θ^n and Θ^k are complicated functions of equilibrium variables:

$$\begin{aligned}
\Theta^n \equiv & \frac{f_N N}{\left(1-\tau^n\right)\Phi}\left\{\left(1-\tau_R^k\right) f_K^2 f_N K\left[\left(1-\tau^n\right)\left(U_{cc}\left(U_c - V_c\right) + \tau_R^k\left(V_{cc} - U_{cc}\right) U_c\right) - \left(1-\tau_R^k\right)\tau^n \kappa U_{cc} U_c\right]\right. \\
& \left. + f_N\left[\left(1-\tau^n\right)\left(V_c - U_c\right) + \tau^n \kappa U_c\right]\left[\left(1-\tau_R^k\right) f_{KN} N U_c - K u_{cc}^0\right] + \left(1-\tau_R^k\right)\tau_R^k f_{KN} f_K K \kappa U_c^2\right\} \\
\Theta^k \equiv & \frac{f_K K U_c}{\Phi}\left\{f_N f_{KN} N\left[\left(1-\tau^n\right)\left(V_c - U_c\right) + \tau^n \kappa U_c\right] + \tau_R^k f_K\left(f_N + f_{KN} K \kappa\right) U_c\right\}
\end{aligned}$$

where

$$\begin{aligned}
U_c & \equiv \beta\left[\pi u_c\left(c_L, n_L\right) + (1-\pi) u_c\left(c_H, n_H\right)\right], \quad U_{cc} \equiv \beta\left[\pi u_{cc}\left(c_L, n_L\right) + (1-\pi) u_{cc}\left(c_H, n_H\right)\right], \\
V_c & \equiv \beta\left[\pi u_c\left(c_L, n_L\right)\frac{e_L n_L}{N} + (1-\pi) u_c\left(c_H, n_H\right)\frac{e_H n_H}{N}\right], \\
V_{cc} & \equiv \beta\left[\pi u_{cc}\left(c_L, n_L\right)\frac{e_L n_L}{N} + (1-\pi) u_{cc}\left(c_H, n_H\right)\frac{e_H n_H}{N}\right], \\
\Phi & \equiv \left(1-\tau_R^k\right)\left(f_K f_N f_{KN} K N\left(\left(1-\tau^n\right)\left(V_{cc} - U_{cc}\right) + \tau^n \kappa U_{cc}\right) + \left(f_N + f_{KN} K \kappa\right) f_K^2 K U_{cc} - f_N f_{KN} N U_c\right) \\
& + \left(f_N + f_{KN} K \kappa\right) K u_{cc}^0
\end{aligned}$$

Then, we have the following lemma

Lemma 2 *In equilibrium $n_H > n_L$ and $u_c\left(c_L, n_L\right) > u_c\left(c_H, n_H\right)$.*

Proof. Let $\tilde{w} \equiv \left(1-\tau^n\right) w$. The intratemporal condition of an agent with productivity e is

$$u_n\left(c, n\right) + \tilde{w} e u_c\left(c, n\right) = 0,$$

[1] Mathematica codes that automatically calculate all the algebra for this section can be found in our websites: http://www.dyrda.info/ or http://sites.google.com/site/marcelozouainpedroni/

where

$$c = \tilde{w}en + \left(1-\tau_R^k\right)(1+r)\,a + T.$$

It follows from the implicit function theorem that

$$\frac{dn}{de} = -\frac{\tilde{w}u_c + \tilde{w}^2 enu_{cc} + \tilde{w}nu_{nc}}{(\tilde{w}e)^2 u_{cc} + 2\tilde{w}eu_{cn} + u_{nn}}$$

Using Assumption 1 this simplifies to

$$\frac{dn}{de} = \frac{\kappa n}{e} > 0,$$

which establishes the first result. Next, notice that

$$\frac{dc}{de} = \tilde{w}n + \tilde{w}e\frac{dn}{de}.$$

Taking the derivative of $u_c(c,n)$ with respect to e and using the last two equations it follows that

$$\frac{du_c(c,n)}{de} = u_{cc}\tilde{w}n + (u_{cc}\tilde{w}e + u_{cn})\,\frac{\kappa n}{e}.$$

Finally, using Assumption 1 we obtain

$$\frac{du_c(c,n)}{de} = u_{cc}\tilde{w}n < 0,$$

which establishes the second result. ■

Proposition 1 *In the uncertainty economy, if u satisfies Assumption A, then, the optimal tax system is such that* $\tau^k = 0$,

$$\tau^n = \frac{(\nu-1)\,\pi\,(1-\pi)\,(e_H n_H - e_L n_L)}{(\nu-1)\,\pi\,(1-\pi)\,(e_H n_H - e_L n_L) + \kappa N\,(\pi\nu + (1-\pi))} > 0, \quad \textit{where } \nu \equiv \frac{u_c(c_L,n_L)}{u_c(c_H,n_H)}, \tag{1.7}$$

and $T < 0$ balances the budget.

Proof. First notice that the optimal tax system must satisfy $\Theta^n = 0$ and $\Theta^k = 0$, otherwise there would exist variations in $\left(\tau^n,\tau_R^k\right) \in (-\infty,1)^2$ that would increase welfare. $\Theta^k = 0$ simplifies to $\theta_1^k + \theta_2^k\tau^n + \theta_3^k\tau_R^k = 0$ where

$$\theta_1^k \equiv f_N f_{KN} N\,(V_c - U_c)\,,\quad \theta_2^k \equiv f_N f_{KN} N\,((1+\kappa)\,U_c - V_c)\,,\quad \text{and}\quad \theta_3^k \equiv f_K\,(f_N + f_{KN}K\kappa)\,U_c.$$

Solving this equation for τ_R^k leads to

$$\tau_R^k = -\frac{f_N f_{KN} N\left[\left(V_c - U_c\right) + \left(\left(1+\kappa\right)U_c - V_c\right)\tau^n\right]}{f_K\left(f_N + f_{KN}K\kappa\right)U_c}.$$

Substituting this into $\Theta^n = 0$ and simplifying (for these steps we refer once again to the Mathematica codes) entails

$$\tau^n = \frac{V_c - U_c}{V_c - \left(1+\kappa\right)U_c}.$$

Substituting this back in the equation for τ_R^k we obtain $\tau_R^k = 0$; and, therefore, $\tau^k = 0$. This is the only pair $\left(\tau^n, \tau_R^k\right) \in (-\infty, 1)^2$ that solves the system $\Theta^n = 0$ and $\Theta^k = 0$. The fact that the optimal level of $\tau^n > 0$ follows from Lemma 2. To obtain equation (1.7), first notice that, using the definitions of U_c and V_c above we obtain

$$\tau^n = \frac{\beta\left(\pi u_c^L e_L n_L + \left(1-\pi\right)u_c^H e_H n_H\right) - \beta N\left(\pi u_c^L + \left(1-\pi\right)u_c^H\right)}{\beta\left(\pi u_c^L e_L n_L + \left(1-\pi\right)u_c^H e_H n_H\right) - \left(1+\kappa\right)\beta N\left(\pi u_c^L + \left(1-\pi\right)u_c^H\right)}.$$

Then, equation (1.7) follows from simplifying this using the definitions of ν and N. ■

1.2 Detailed proof for inequality economy

Again we start with the definition of a competitive equilibrium:

Definition 2 *A **tax distorted competitive equilibrium** is* $\left(a_L, a_H, n_L, n_H, r, w; \tau^n, \tau^k, T\right)$ *such that*

1. *For* $i \in \{L, H\}$, (a_i, n_i) *solves*

$$\max_{a_i, n_i} u\left(\omega_i - a_i, \bar{n}\right) + \beta u\left(c_i, n_i\right), \quad \text{s.t. } c_i = \left(1-\tau^n\right)wn_i + \left(1 + \left(1-\tau^k\right)r\right)a_i + T;$$

2. $r = f_K(K, N)$, $w = f_N(K, N)$, *where* $K = pa_L + (1-p)a_H$ *and* $N = pn_L + (1-p)n_H$*;*

3. *and,* $\tau^n wN + \tau^k rK = G + T$.

Seven equations determine such an equilibrium. The first order conditions of the agent's problem,

$$\begin{aligned}
u_c\left(\omega_L - a_L, \bar{n}\right) &= \beta\left(1-\tau_R^k\right)\left(1+r\right)u_c\left(c_L, n_L\right), &(1.8)\\
u_c\left(\omega_H - a_H, \bar{n}\right) &= \beta\left(1-\tau_R^k\right)\left(1+r\right)u_c\left(c_H, n_H\right), &(1.9)\\
u_n\left(c_L, n_L\right) &= -\left(1-\tau^n\right)we_L u_c\left(c_L, n_L\right), &(1.10)\\
u_n\left(c_H, n_H\right) &= -\left(1-\tau^n\right)we_H u_c\left(c_H, n_H\right); &(1.11)
\end{aligned}$$

the first order conditions of the firm's problem, (1.4) and (1.5); and the government's budget constraint, (1.6).

Using equations (1.4) and (1.5) to substitute out for r and w we are left with a system of five equations that any vector $\left(a_L, a_H, n_L, n_H, \tau^n, \tau_R^k, T\right)$ of equilibrium values must satisfy. Letting the welfare function be utilitarian, that is

$$W \equiv pU_L + (1-p)\, U_H$$

where

$$U_i \equiv u\left(\omega_i - a_i, \bar{n}\right) + \beta u\left(c_i, n_i\right),$$

and totally differentiating the five equilibrium equations together with this definition and making the appropriate simplifications using Assumption 1 we obtain the following equation:

$$dW = \Psi^n d\tau^n + \Psi^k d\tau_R^k.$$

The definitions of Ψ^n and Ψ^k can be found in the Mathematica code.

Lemma 3 *Under Assumption 1, in equilibrium $n_H = n_L$. Further, if u displays CARA, then $a_H - a_L = \left(\omega_H - \omega_L\right) / \left(1 + \left(1 - \tau_R^k\right)(1+r)\right)$, if u is GHH then*
$a_H - a_L = \left(\omega_H - \omega_L\right) / \left(1 + \beta^{-\frac{1}{\sigma}} \left(\left(1 - \tau_R^k\right)(1+r)\right)^{\frac{\sigma-1}{\sigma}}\right)$, *and in either case $u_c\left(c_L, n_L\right) > u_c\left(c_H, n_H\right)$.*

Proof. Let $\tilde{w} \equiv (1-\tau^n)\, w$ and $\tilde{R} \equiv \left(1 - \tau_R^k\right)(1+r)$. The intratemporal condition of an agent with initial endowment of ω is

$$u_n(c, n) + \tilde{w} e u_c(c, n) = 0,$$

where

$$c = \tilde{w} n + \tilde{R} a + T,$$

and it follows that

$$\frac{dc}{d\omega} = \tilde{R}\frac{da}{d\omega}$$

Using the implicit function theorem we obtain

$$\frac{dn}{d\omega} = -\frac{\left(u_{nc} + \tilde{w} u_{cc}\right) \tilde{R} \frac{da}{d\omega}}{\left(\tilde{w} e\right)^2 u_{cc} + 2\tilde{w} e u_{cn} + u_{nn}},$$

and from Assumption 1 it follows that $dn/d\omega = 0$, which establishes the first result. Taking the derivative of $u_c(c, n)$ with respect to ω and using the last two equations it follows that

$$\frac{du_c(c, n)}{d\omega} = u_{cc}\tilde{R}\frac{da}{d\omega},$$

and $u_c\left(c_L, n_L\right) > u_c\left(c_H, n_H\right)$ as long as $da/d\omega > 0$.

If u displays CARA, then $u_{cc} = -\psi u_c$. From the intertemporal condition we have

$$u_c\left(\omega - a, n_0\right) = \beta \tilde{R} u_c\left(\tilde{w} n + \tilde{R} a + T, n\right),$$

and the implicit function theorem entails

$$\frac{da}{d\omega} = \frac{u_{cc}^0}{u_{cc}^0 + \beta \tilde{R}^2 u_{cc}^1}.$$

Using the CARA assumption this simplifies to

$$\frac{da}{d\omega} = \frac{1}{1 + \tilde{R}} > 0.$$

If u is GHH, then the intertemporal condition implies

$$\left(\omega - a - \chi \frac{\bar{n}^{1+\frac{1}{\kappa}}}{1 + \frac{1}{\kappa}}\right)^{-\sigma} = \beta \tilde{R}\left(\tilde{w} n + \tilde{R} a + T - \chi \frac{n^{1+\frac{1}{\kappa}}}{1 + \frac{1}{\kappa}}\right)^{-\sigma},$$

and another application of the implicit function theorem implies

$$\frac{da}{d\omega} = \frac{1}{1 + \beta^{-\frac{1}{\sigma}} \tilde{R}^{\frac{\sigma-1}{\sigma}}} > 0.$$

■

Proposition 2 *In the inequality economy, if u satisfies Assumption A and has CARA is GHH, then the optimal tax system is such that $\tau^n = 0$,*

$$\tau^k = \frac{\left(\frac{1+r}{r}\right)(\nu - 1)\, p\,(1-p)\,(\omega_H - \omega_L)}{(\nu - 1)\, p\,(1-p)\,(\omega_H - \omega_L) + \frac{\rho}{\psi}\left(p\nu + (1-p)\right)} > 0, \tag{1.12}$$

where $\rho \equiv \frac{2+(1-\tau^k)r}{2+r}$ for CARA, $\rho \equiv \frac{1+\beta^{-\frac{1}{\sigma}}(1+(1-\tau^k)r)^{\frac{\sigma-1}{\sigma}}}{1+r+\beta^{\frac{1}{\sigma}}(1+(1-\tau^k)r)^{\frac{1}{\sigma}}}$ for GHH, and ψ is the level of absolute risk aversion[2]. *$T < 0$ balances the budget.*

Proof. The optimal tax system must satisfy $\Psi^n = 0$ and $\Psi^n = 0$. If u displays CARA, then, it follows that (see Mathematica code for details) $\tau^n = 0$, and

$$\tau_R^k = \frac{(a_2 - a_1)(1 + R)(1 - p)\, p\psi\left(u_c^L - u_c^H\right)}{p\left(1 + (a_2 - a_1)(1 + R)(1 - p)\psi\right)\left(u_c^L - u_c^H\right) + u_c^H}.$$

[2] The level of absolute risk aversion is endogenous is the GHH case and should be evaluated at the aggregate levels of consumption and labor.

Simplifying using the definition of ν, $(a_2 - a_1)$ from Lemma 3, we obtain

$$\tau_R^k = \frac{(\nu-1)\,p\,(1-p)\,(\omega_H-\omega_L)}{(\nu-1)\,p\,(1-p)\,(\omega_H-\omega_L)+\frac{1}{\psi}\frac{1+\left(1-\tau_R^k\right)(1+r)}{2+r}\,(p\nu+(1-p))},$$

The fact that $\tau^k = ((1+r)/r)\,\tau_R^k$, then implies equation (1.12) with $\rho = \frac{2+(1-\tau^k)r}{2+r}$.

If u is GHH, then $\Psi^n = 0$ and $\Psi^n = 0$ implies

$$\tau_R^k = \frac{(\nu-1)\,p\,(1-p)\,(a_2-a_1)}{(\nu-1)\,p\,(1-p)\,(a_2-a_1)+\frac{C-\left(\frac{\kappa}{1+\kappa}\right)wN}{\sigma\left(\left(\beta\left(1-\tau_R^k\right)(1+r)\right)^{\frac{1}{\sigma}}+(1+r)\right)}\,(p\nu+(1-p))}$$

where

$$C = pc_1 + (1-p)\,c_2.$$

Using $(a_2 - a_1)$ from Lemma 3, $\psi = \sigma/\left(C - \frac{\kappa}{1+\kappa}wN\right)$, and the fact that $\tau^k = ((1+r)/r)\,\tau_R^k$, then implies equation (1.12) with $\rho = \frac{1+\beta^{-\frac{1}{\sigma}}(1+(1-\tau^k)r)^{\frac{\sigma-1}{\sigma}}}{1+r+\beta^{\frac{1}{\sigma}}(1+(1-\tau^k)r)^{\frac{1}{\sigma}}}$. ■

1.3 Relationship to Dávila, Hong, Krusell and Ríos-Rull (2012)

The results established in Dávila et al. (2012) have an interesting relationship to the ones we obtain in this paper. We will use the last result to explain this relationship. Among other things, Dávila et al. (2012) show that the competitive equilibrium allocation in the SIM model is constrained inefficient. That is, the incomplete market structure itself induces outcomes that could be improved upon if consumers merely acted differently; if they used the same set of markets but departed from purely self-interested optimization. The constrained inefficiency results from a pecuniary externality. The savings and labor supply decisions of the agents affects the wage and interest rates and, therefore, the uncertainty and inequality in the economy. These effects are not internalized by the agents and inefficiency follows. Notice that the planner's problem in their environment is significantly different from the Ramsey problem described here. There the planner affects allocations directly and prices indirectly, as a result redistribution and insurance can only occur via the manipulation of equilibrium prices. Whereas here the Ramsey planner affects (after tax) prices directly and allocations indirectly.

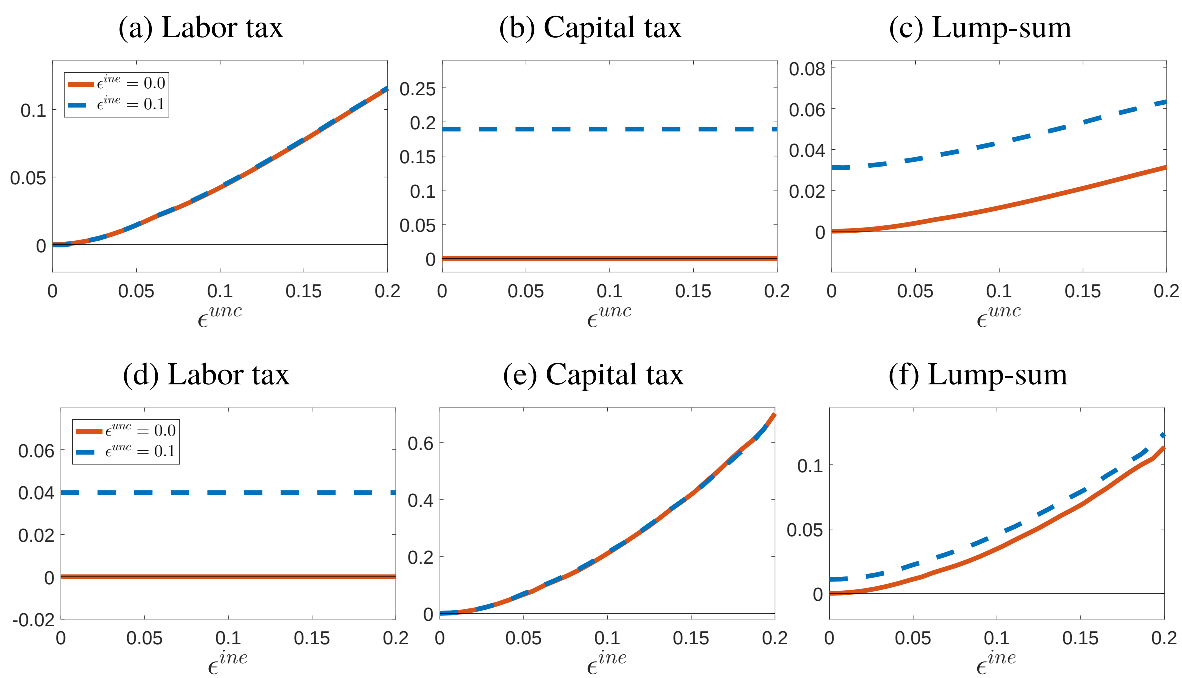
In a setting similar to the inequality economy just described above, for instance, Dávila et al. (2012) show that there is under accumulation of capital. A higher level of capital would decrease interest rates and increase wages, reducing inequality. A naive extrapolation of this logic would suggest that capital taxes should be negative so as to encourage savings. This logic, however, does not take into account the more relevant direct effect of the tax system on after tax prices. Proposition 2 shows that the opposite is true: capital taxes should be positive.

1.4 Uncertainty and inequality

If both uncertainty and inequality are present, the optimal tax system has to balance three objectives: minimize distortions, provide insurance and redistribution. A reasonable conjecture is that under Assumption 1 the optimal tax system will be a convex combination of the ones in Propositions 1 and 2, that is, positive labor and capital income taxes with magnitudes associated with the levels of uncertainty and inequality in the economy. A more subtle conjecture, associated with Assumption 1, is that capital (labor) income taxes should be invariant with respect to the level of uncertainty (inequality). We corroborate these conjectures with a numerical example the results of which are in Figure 1[3].

The first row of Figure 1 shows the optimal tax system with the level of uncertainty (embodied by the parameter ϵ^{unc}) in the x-axis with two levels of inequality: $\epsilon^{ine} = 0$ (solid line) and $\epsilon^{ine} = 0.1$ (dashed line). The solid lines corroborate Proposition 1. The comparison between the dashed and the solid lines corroborates the conjectures made above. The labor tax is increasing with the level of uncertainty and independent on the level of inequality whereas capital taxes increase with the level of inequality and are independent on level of risk. The second row of Figure 1 shows the results for the analogous experiment with ϵ^{ine} on the x-axis and $\epsilon^{unc} = 0$ (solid) and $\epsilon^{unc} = 0.1$ (dashed).

Figure 1: Optimal taxes in the presence of both uncertainty and inequality.



[3]We use GHH preferences which satisfy Assumption 1. The most relevant interpretation of this two-period economy is that each period corresponds to half of the working life of a person. Accordingly, we set $\beta = 0.95^{20}$ and $\delta = 1 - 0.9^{20}$. Other parameters are set to satisfy the usual targets: $\sigma = 2$, $\kappa = 0.72$, $\chi = 6$, $\bar{n} = 0.3$, $\omega = 3.5$, $\pi = p = 0.5$, and $f(K, N) = K^{\alpha} N^{1-\alpha} - \delta K$ with $\alpha = 0.36$. G is set to 0, but any other feasible level would just shift the lump-sum transfers correspondingly.

2 Derivation of the final tax in transition

This section of the Appendix provides the derivation of the final level of labor tax that balances the government's budget constraint,

$$G + r_t B_t = B_{t+1} - B_t + \tau_t^c C_t + \tau_t^n w_t N_t + \tau_t^k r_t \left(\hat{A}_t + K_t + B_t - A_t \right) - T_t.$$

We assume that the budget is balanced if government debt is bounded. Manipulating the equation above we obtain

$$B_{t+1} = \left(1 + \left(1 - \tau_t^k\right) r_t\right) B_t + G + T_t - \tau_t^k r_t \left(K_t + \hat{A}_t - A_t \right) - \tau_t^n w_t N_t - \tau_t^c C_t.$$

Next, define

$$R_t \equiv 1 + \left(1 - \tau_t^k\right) r_t, \quad \text{and} \quad D_t \equiv G + T_t - \tau_t^k r_t \left(K_t + \hat{A}_t - A_t \right) - \tau_t^n w_t N_t - \tau_t^c C_t.$$

It follows that

$$B_{t+1} = R_t B_t + D_t \tag{2.1}$$

Iterating this equation forward we obtain

$$B_t = \left(\prod_{i=1}^{t-1} R_i \right) B_1 + \sum_{j=1}^{t-2} \left(\prod_{i=j+1}^{t-1} R_i \right) D_j + D_{t-1}$$

That is, given debt at $t = 1$, debt at t is given by the present value of B_1, plus the accumulated deficits in present value.

Now, we compute the path $\{B_t\}_{t=1}^{t^*}$. Let $t^* + 1$ be the period in which taxes become constant (i.e. taxes are set to their final levels at $t^* + 1$). Suppose the paths $\left\{\tau_t^k, \tau_t^n, \tau_t^c, T_t\right\}_{t=1}^{t^*}$ are given. First, we calculate the debt at period $t^* + 1$ associated with these paths for taxes,

$$B_{t^*+1} = \left(\prod_{i=1}^{t^*} R_i \right) B_1 + \sum_{j=1}^{t^*-1} \left(\prod_{i=j+1}^{t^*} R_i \right) D_j + D_{t^*}.$$

To compute the debt levels for $t \in \{1, ..., t^*\}$ we can use equation (2.1) to solve for it backwards,

$$B_t = \frac{B_{t+1} - D_t}{R_t}.$$

Finally, we can compute the final labor tax and $\{B_t\}_{t=t^*+2}^{\bar{t}}$. Suppose the paths $\left\{\tau_t^k, \tau_t^c, T_t\right\}_{t=t^*+1}^{\bar{t}}$ are given and constant over time. We solve for τ^n that implies $B_{\bar{t}} = B_{\bar{t}-1}$ where $\bar{t}$ is a very large number. We start by computing $B_{\bar{t}-1}$ taking as given B_{t^*+1} and the constant final level of taxes τ^k, τ^n, τ^c and T,

$$B_{\bar{t}-1} = \left(\prod_{i=t^*+1}^{\bar{t}-2} R_i\right) B_{t^*+1} + \sum_{j=t^*+1}^{\bar{t}-3} \left(\prod_{i=j+1}^{\bar{t}-2} R_i\right) D_j + D_{\bar{t}-2},$$

Using the definition for D_t we obtain

$$B_{\bar{t}-1} = \Psi - \tau^n \Omega. \tag{2.2}$$

where

$$\begin{aligned}
\Psi \equiv & \left(\prod_{i=t^*+1}^{\bar{t}-2} R_i\right) B_{t^*+1} + \sum_{j=t^*+1}^{\bar{t}-3} \left(\prod_{i=j+1}^{\bar{t}-2} R_i\right) \left(G - \tau^k r_j \left(K_j + \hat{A}_j - A_j\right) - \tau^c C_j\right) \\
& + G - \tau^k r_{\bar{t}-2} K_{\bar{t}-2} - \tau^c C_{\bar{t}-2}, \\
\Omega \equiv & \sum_{j=t^*+1}^{\bar{t}-3} \left(\prod_{i=j+1}^{\bar{t}-2} R_i\right) (w_j N_j) + w_{\bar{t}-2} N_{\bar{t}-2}.
\end{aligned}$$

Substituting this and $B_{\bar{t}} = B_{\bar{t}-1}$ on (2.1) evaluated at $\bar{t}-1$ we obtain:

$$(\Psi - \tau^n \Omega) = R_{\bar{t}-1} (\Psi - \tau^n \Omega) + D_{\bar{t}-1},$$

using the definition of D_t,

$$(\Psi - \tau^n \Omega) = R_{\bar{t}-1} (\Psi - \tau^n \Omega) + G - \tau^k r_{\bar{t}-1} \left(K_{\bar{t}-1} + \hat{A}_{\bar{t}-1} - A_{\bar{t}-1}\right) - \tau^n w_{\bar{t}-1} N_{\bar{t}-1} - \tau^c C_{\bar{t}-1},$$

and therefore

$$\tau^n = \frac{(R_{\bar{t}-1} - 1)\,\Psi + G - \tau^k r_{\bar{t}-1} \left(K_{\bar{t}-1} + \hat{A}_{\bar{t}-1} - A_{\bar{t}-1}\right) - \tau^c C_{\bar{t}-1}}{w_{\bar{t}-1} N_{\bar{t}-1} + (R_{\bar{t}-1} - 1)\,\Omega}.$$

Then, $B_{\bar{t}-1}$ is given by 2.2 and we can solve for $\{B_t\}_{t=t^*+2}^{\bar{t}-2}$ backwards using (2.1).

3 Algorithms

Here we describe the algorithms used to obtain our results.

3.1 Transition

Algorithm for computing the transition between steady states[4]

1. Solve for the initial stationary equilibrium.

2. Assume the economy converges to a new stationary equilibrium in $\bar{t}$ periods and guess a sequence $K_2, ..., K_{\bar{t}-1}$.

3. Solve for the new tax on labor such that given $K_2, ..., K_{\bar{t}-1}$ and the path for the other taxes, government debt is unchanged between $\bar{t}-1$ and $\bar{t}$. Compute the associated path for the government debt, $B_1, ..., B_{\bar{t}-1}$ (for details see Section 2).

4. Solve for the final stationary equilibrium given final tax rates τ^k, τ^n, τ^c and T, and $B_{\bar{t}}$. Compute $K_{\bar{t}}$.

5. Solve for households savings decisions in transition.

6. Update the path of capital, i.e. take the initial stationary distribution over wealth and productivity and use the decision rules computed above to simulate the economy forward. Then, check for market clearing at each date and adjust $K_2, ..., K_{\bar{t}-1}$ appropriately.

7. If the new sequence for capital is the close to the old, we have found the equilibrium path. Otherwise go back to step 5.

8. Increase $\bar{t}$ until the solution stops changing.

3.2 Global Optimization

The global optimization algorithm is an application to our problem of the procedure (with some adjustments) described in Kucherenko and Sytsko (2005) and Guvenen (2011). The algorithm is as follows:

1. Initialization

   (a) Determine the bounds $[b^0_{\text{min}}(n), b^0_{\text{max}}(n)]$ for each fiscal instrument $n \in N$.

   (b) Generate a matrix of policies of dimension $N \times I \times G_{\text{max}}$ with the use of a quasi-random low-discrepancy sequence known as Sobol, where I is the number of function evaluations at each global stage and G_{max} is the maximum number of global iterations.

[4] This is an extension of the procedure proposed by Domeij and Heathcote (2004). To solve for agent's decision rules we use the endogenous grid method (see Carroll (2006)).

(c) Let P_g be the $N \times I$ matrix of policies associated with global iteration g. Set the global iteration $g = 1$. Set the number of local maxima $NLM = 1$.

2. Global stage (pre-testing): for each $i \leq I$ do the following steps:

   (a) If the library (see part (e)) is non-empty, find within it the already computed transition that has the policy vector, $P_g^*(\cdot, i)$, closest to $P_g(\cdot, i)$.

   (b) Use the previously saved path for capital associated with $P_g^*(\cdot, i)$ as an initial guess to compute the transitional dynamics associated with policy vector $P_g(\cdot, i)$. If the library is empty, use as an initial condition an interpolation between the initial and final stationary levels of capital.

   (c) Evaluate welfare over transition.

   (d) Save the welfare gain/loss at $W(i)$ position of the welfare vector W.

   (e) Save $P_g(\cdot, i)$ and its associated equilibrium path for capital into a library of initial conditions.

3. Local stage:

   (a) Organize vector W in the ascending order and the vector of policies P_g accordingly.

   (b) Define a reduced sample set P_R of size $N \times I_R$, where $I_R \leq I$, i.e. $P_R \equiv \{P_g(\cdot, i) : i \leq I_R\}$.

   (c) For each policy vector in $i \leq I_R$ run the local solver BOBYQA[5] i.e. search for the welfare maximizing policy starting from policy $P_R(\cdot, i)$.

   (d) Denote by W_R to be the vector of dimension I_R of welfare gains/losses for the reduced sample of policies. Save the policy and welfare associated with it at $P_R(\cdot, i)$ and $W_R(i)$.

   (e) Organize vector W_R in the ascending order and the vector of policies P_R accordingly.

4. Update the set of local maxima:

   (a) If $g = 1$ then for each k, l where $k \neq l$ and $k, l \leq I_R$ check

$$\|P_R(\cdot, k) - P_R(\cdot, l)\| > \varepsilon_{LM}. \tag{3.1}$$

   If condition (3.1) holds then we call the two local maxima distinct and set: $NLM = NLM + 1$, $P_{LM}(\cdot, NLM) = P_R(\cdot, k)$ and $W_{LM}(NLM) = W_R(k)$, where P_{LM} and W_{LM} are accordingly a matrix of policies and associated vector of welfare gains/losses.

[5]See Powell (2009). The parameters for BOBYQA were: RHOBEG= 10^{-1}, RHOEND= 10^{-4}, MAXITE= 450.

(b) If $g > 1$ then for each k, l where $k \neq l$ and $k, l \leq I_R$ check

$$\|P_R(\cdot, k) - P_R(\cdot, l)\| > \varepsilon_{LM}, \tag{3.2}$$

and for each $k \leq I_R$ and $j \leq NLM$ check

$$\|P_R(\cdot, k) - P_{LM}(\cdot, j)\| > \varepsilon_{LM}. \tag{3.3}$$

If conditions (3.2) and (3.3) are satisfied, set $NLM = NLM + 1$, $P_{LM}(\cdot, NLM) = P_R(\cdot, k)$, and $W_{LM}(NLM) = W_R(k)$.

5. Adjust the bounds:

(a) For each $n \in N$ compute the following auxiliary variables

$$\begin{aligned} X_{\text{max}}(n) &= \max_{i \in \{1,\ldots,\min(NLM, NLM_b)\}} \left\{ b^0_{\text{min}}(n) + P_{LM}(n, i) \times \left(b^0_{\text{max}}(n) - b^0_{\text{min}}(n) \right) \right\} \\ X_{\text{min}}(n) &= \min_{i \in \{1,\ldots,\min(NLM, NLM_b)\}} \left\{ b^0_{\text{min}}(n) + P_{LM}(n, i) \times \left(b^0_{\text{max}}(n) - b^0_{\text{min}}(n) \right) \right\} \end{aligned}$$

where NLM_b is the upper bound on the number of local minima used in the adjustment of the bounds.

(b) For each $n \in N$, adjust the bounds as follows

(i) If $(X_{\text{max}}(n) - X_{\text{min}}(n)) > \alpha_1 (b^0_{\text{max}}(n) - b^0_{\text{min}}(n))$ then

$$\begin{aligned} b^1_{\text{max}}(n) &= X_{\text{max}}(n) + \theta_1 (X_{\text{max}}(n) - X_{\text{min}}(n)) \\ b^1_{\text{min}}(n) &= X_{\text{min}}(n) - \theta_1 (X_{\text{max}}(n) - X_{\text{min}}(n)) \\ stop(n) &= 0 \end{aligned}$$

(ii) If $X_{\text{min}}(n) > (b^0_{\text{max}}(n) - \alpha_2 (b^0_{\text{max}}(n) - b^0_{\text{min}}(n)))$ then

$$\begin{aligned} b^1_{\text{max}}(n) &= b^0_{\text{max}}(n) + \theta_2 \left(b^0_{\text{max}}(n) - b^0_{\text{min}}(n) \right) \\ b^1_{\text{min}}(n) &= b^0_{\text{max}}(n) - \theta_3 \left(b^0_{\text{max}}(n) - b^0_{\text{min}}(n) \right) \\ stop(n) &= 0 \end{aligned}$$

(iii) If $X_{\text{max}}(n) < (b^0_{\text{min}}(n) + \alpha_2 (b^0_{\text{max}}(n) - b^0_{\text{min}}(n)))$ then

$$\begin{aligned} b^1_{\text{max}}(n) &= b^0_{\text{min}}(n) + \theta_3 \left(b^0_{\text{max}}(n) - b^0_{\text{min}}(n) \right) \\ b^1_{\text{min}}(n) &= b^0_{\text{min}}(n) - \theta_2 \left(b^0_{\text{max}}(n) - b^0_{\text{min}}(n) \right) \end{aligned}$$

$$stop(n) = 0$$

(iv) Else

$$\begin{aligned} b^1_{\max}(n) &= X_{\max}(n) + \theta_4 \left(b^0_{\max}(n) - b^0_{\min}(n)\right) \\ b^1_{\min}(n) &= X_{\min}(n) - \theta_4 \left(b^0_{\max}(n) - b^0_{\min}(n)\right) \\ stop(n) &= 1 \end{aligned}$$

6. Stopping rule: if $stop(n) = 1$ for all $n \in N$, then, let NLM_g be the number of local minima found in the current global iteration and compute

$$NLM_{exp} = \frac{NLM_g(I_R - 1)}{I_R - NLM_g - 2}$$

provided that $I_R > NLM_g + 2$. If $NLM_{exp} < NLM_g + 0.5$ then STOP and go to Step 7[6]. Otherwise set the global iteration to $g = g + 1$, use bound updates i.e. set $b^0_{\max} = b^1_{\max}$, $b^0_{\min} = b^1_{\min}$ and go to Step 2.

7. Pick the global optimum, i.e.

$$\begin{aligned} j_{\max} &= \arg\max_{j \in NLM} W_R(j) \\ P_{GM}(\cdot) &= P_{LM}(\cdot, j_{\max}) \end{aligned}$$

In the computational implementation of the algorithm presented above we have to impose the values of the following parameters: (1) number of fiscal instruments N (ii) initial bounds on the instruments $\left[b^0_{n,\min}, b^0_{n,\max}\right]$ (iii) number of function evaluations in the global stage I (iv) maximum number of global iterations $G_{\max}$ (5) the size of the reduced sample I_R (6) the distance separating two local maxima ε_{LM} (7) bounds adjustment parameters $\{NLM_b, \alpha_1, \alpha_2, \theta_1, \theta_2, \theta_3, \theta_4\}$ (8) Stopping tolerance for the bounds ε_B. In the main experiment of the paper we set $N = 15$ - see the detailed description of the main experiment in Section 3.2 of the paper.

The number of function evaluations in the global stage is set to $147,456$, which is the multiple of 1152, the number of cores we use in the computational implementation of the algorithm (so that in each global iteration each core conducts 128 evaluations of welfare over transition). Further, we set the maximum number of global iterations, $G_{\max}$, to 8, and in numerous robustness checks we have not hit this bound. The size of the reduced sample, I_R, is set to 1152, so that each core conducts one local search. The distance separating two local maxima is set to

$$\varepsilon_{LM} \equiv \sqrt{\left(\sum_{n=1}^{N} (b^0_{max}(n) - b^0_{min}(n))^2\right)},$$

[6]This is a Baysian rule, see Guvenen (2011) for an heuristic explanation for it.

where $b^0_{max}(n)$ and $b^0_{min}(n)$ are the initial bounds set in step 1. The bounds adjustment parameters are set to the following values: $NLM_b = 8$, $\alpha_1 = 0.5$, $\alpha_2 = 0.1$, $\theta_1 = 0.2$, $\theta_2 = 0.7$, $\theta_3 = 0.1$, $\theta_4 = 0.15$. The idea behind the adjustment of the bounds in step 5 part (b) is the following. Fix a particular $n \in N$, then

(i) If the local maxima are distributed somewhat evenly over the bounds, i.e. $(X_{\text{max}}(n) - X_{\text{min}}(n)) < 0.5 = \alpha_1$, then we increase the bounds on both sides by 20 percent (θ_1) of $(X_{\text{max}}(n) - X_{\text{min}}(n))$.

(ii) If the local maxima are bunched up in the top 10 percent (α_2) of the the bounds, then we increase the upper and lower bounds in proportion to $(b^0_{\text{max}}(n) - b^0_{\text{min}}(n))$ using the parameters θ_2, and θ_3.

(iii) If the local maxima are bunched up in the bottom 10 percent (α_2) of the the bounds, then we decrease the upper and lower bounds in proportion to $(b^0_{\text{max}}(n) - b^0_{\text{min}}(n))$ using the parameters θ_2, and θ_3.

(iv) If none of the other conditions are satisfied, that is, it the local maxima are bunched up in the middle of the bounds we reduce the upper bound and increase the lower bound by by 15 percent (θ_4).

We also conducted robustness checks with regards to the bound adjustment procedure and concluded that the values of these parameters affect just the pace of convergence rather than the results of the global optimization. The algorithm is implemented using MPI library and our experiments were conducted at the Mesabi cluster at the Minnesota Supercomputing Institute.

4 Results for the complete markets economies

4.1 Environment

Consider an economy populated by a continuum of infinitely-lived agents divided into types $i \in I$ of size π_i. Each agent of type $i \in I$ ranks streams of consumption and hours worked $\{c_{i,t}, n_{i,t}\}$ according to the preferences

$$\sum_{t=0}^{\infty} \beta^t u\left(c_{i,t}, n_{i,t}\right), \tag{4.1}$$

with period utility function given by

$$u\left(c, n\right) = \frac{1}{1-\sigma}\left(c - \chi \frac{n^{1+\frac{1}{\kappa}}}{\left(1+\frac{1}{\kappa}\right)}\right)^{1-\sigma}$$

where σ is the inverse of the intertemporal elasticity of substitution, κ is the Frisch elasticity of labor supply and χ is the weight on the disutility from hours worked. An agent of type $i \in I$ with productivity e_i works $n_{i,t}$ each period. Aggregates are denoted without the subscript i: $C_t = \sum_i \pi_i c_{i,t}$, $N_t = \sum_i \pi_i e_i n_{i,t}$ and $K_t = \sum_i \pi_i k_{i,t}$.

Consumption-capital good is produced with a concave, constant returns to scale technology, $F\left(K, N\right)$, that uses aggregate capital, K, and aggregate labor, N. Thus, the resource constraint of the economy is given by

$$C_t + G_t + K_{t+1} = F\left(K_t, N_t\right) + \left(1-\delta\right) K_t, \quad \text{for } t \geq 0 \tag{4.2}$$

where $\{G_t\}_{t=0}^{\infty}$ is an exogenous sequence of government spending and δ is the rate of depreciation of the capital stock.

4.1.1 Agent's problem

Let p_t denote the price of the consumption good in period t in terms of consumption in period 0, w_t and r_t denote the real wage and the rental rate of capital in period t. Let $b_{i,t}$ and $k_{i,t}$ denote the number of units of government debt and capital held between periods $t-1$ and t, and R_t denote its gross return (between $t-1$ and t). Given $k_{i,0}$, $b_{i,0}$, prices $\{p_{t,} w_t, r_t\}_{t=0}^{\infty}$ and policies $\left\{\tau_t^n, \tau_t^k, T_t\right\}_{t=0}^{\infty}$, the agent chooses $\{c_{i,t}, n_{i,t}, k_{i,t+1}, b_{i,t+1}\}$ to maximize (1) subject to the intertemporal budget constraint

$$\sum_{t=0}^{\infty} p_t \left(\left(1-\tau^c\right) c_{i,t} + k_{i,t+1} + b_{i,t+1}\right) \leq \sum_{t=0}^{\infty} p_t \left(\left(1-\tau_t^n\right) w_t e_i n_{i,t} + R_t \left(k_{i,t} + b_{i,t}\right) + T_t\right),$$

where $R_t \equiv 1 + \left(1 - \tau_t^k\right)(r_t - \delta)$, for $t \geq 0$. Since $p_t = R_{t+1}p_{t+1}$, and defining $T \equiv \sum_{t=0}^{\infty} p_t T_t$, this is equivalent to

$$\sum_{t=0}^{\infty} p_t \left(\left(1 - \tau^c\right) c_{i,t} - \left(1 - \tau_t^n\right) w_t e_i n_{i,t}\right) \leq R_0 a_{i,0} + T, \tag{4.3}$$

where $a_{i,0} = k_{i,0} + b_{i,0}$. The first order conditions of agent i's problem are:

$$\begin{aligned}
c_{i,0} &: \frac{u_c\left(c_{i,0}, n_{i,t}\right)}{\left(1 - \tau^c\right) p_0} = \gamma, \quad \forall\, t \geq 0 \\
c_{i,t} &: \frac{\beta^t u_c\left(c_{i,t}, n_{i,t}\right)}{\left(1 - \tau^c\right) p_t} = \gamma, \quad \forall\, t \geq 0 \\
n_{i,t} &: \beta^t u_n\left(c_{i,t}, n_{i,t}\right) = -\gamma p_t \left(1 - \tau_t^n\right) w_t e_i, \quad \forall\, t \geq 0
\end{aligned}$$

thus, in particular,

$$p_t = \beta^t p_0 \frac{u_c\left(c_{i,t}, n_{i,t}\right)}{u_c\left(c_{i,0}, n_{i,0}\right)}, \quad \forall\, t \geq 0 \tag{4.4}$$

$$\frac{u_n\left(c_{i,t}, n_{i,t}\right)}{u_c\left(c_{i,t}, n_{i,t}\right)} = -e_i w_t \frac{\left(1 - \tau_t^n\right)}{\left(1 - \tau^c\right)}, \quad \forall\, t \geq 0, \tag{4.5}$$

which holds across all agents.

4.1.2 Firm's problem

The first order conditions for the firm problem are:

$$r_t = F_k\left(K_t, N_t\right), \quad \forall\, t \geq 0, \tag{4.6}$$

$$w_t = F_n\left(K_t, N_t\right), \quad \forall\, t \geq 0. \tag{4.7}$$

4.1.3 Government's budget constraint

Each period the government finances the expenses G_t and lump sum transfers T_t with proportional income taxes on capital τ_t^k and labor τ_t^n. The government's intertemporal budget constraint is

$$\sum_t p_t \left(G_t + R_t B_t + T_t\right) = \sum_t p_t \left(\tau_t^n w_t N_t + \tau_t^k \left(r_t - \delta\right) K_t + B_{t+1}\right),$$

which is equivalent to

$$R_0 B_0 + T + \sum_t p_t G_t = \sum_t p_t \left(\tau_t^n w_t N_t + \tau_t^k \left(r_t - \delta\right) K_t\right). \tag{4.8}$$

4.1.4 Government's budget constraint

Moreover, notice that τ_0^k and T_0 have not been substituted out in the implementability constraint. The fact that τ_0^n is given together with the equilibrium condition $(1-\tau_0^n)\,w_0 = -U_N^m(0)/U_C^m(0)$ is equivalent to

$$N_0 = \bar{N}_0, \tag{4.9}$$

where $\bar{N}_0$ is defined implicitly as a function of variables given to the Ramsey planner,

$$(1-\tau_0^n)\,f_N\left(K_0, \bar{N}_0\right) = \Omega^n \chi \left(\bar{N}_0\right)^{\frac{1}{\kappa}}.$$

4.1.5 Competitive equilibrium

Definition Given $\{a_{i,0}\}$, K_0, B_0 and $\left(\tau_0^n, \tau_0^k, T_0\right)$, a competitive equilibrium is a policy $\left\{\tau_t^n, \tau_t^k, T_t\right\}_{t=1}^{\infty}$, a price system $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and an allocation $\{c_{i,t}, n_{i,t}, K_{t+1}\}_{t=0}^{\infty}$ such that: (i) agents choose $\{c_{i,t}, n_{i,t}\}_{t=0}^{\infty}$ to maximize utility subject to budget constraint (4.3) taking policies and prices (that satisfy $p_t = R_{t+1}p_{t+1}$) as given; (ii) firms maximize profits; (iii) the government's budget constraint (4.8) holds; (iv) markets clear: the resource constraint (4.2) holds.

4.2 A Simple Characterization of Equilibrium

Let $\varphi \equiv \{\varphi_i\}$ be the market weights normalized so that $\sum_i \varphi_i = 1$ with $\varphi_i \geq 0$. Then, given aggregate levels C_t and N_t, the individual levels can be found by solving the following static subproblem for each period t:

$$U\left(C_t, N_t; \varphi\right) \equiv \max_{c_{i,t}, n_{i,t}} \sum_i \pi_i \varphi_i u\left(c_{i,t}, n_{i,t}\right) \quad \text{s.t.} \sum_i \pi_i c_{i,t} = C_t \quad \text{and} \quad \sum_i \pi_i e_i n_{i,t} = N_t \tag{4.10}$$

In what follows we obtain a simple formula for "representative agent" indirect utility $U\left(C_t, N_t; \varphi\right)$.

The Lagrangian for this problem, using the parametrization for the utility function is

$$L = \sum_i \pi_i \varphi_i \left[\frac{1}{1-\sigma}\left(c_{i,t} - \chi\frac{\kappa}{1+\kappa}\left(n_{i,t}\right)^{1+\frac{1}{\kappa}}\right)^{1-\sigma}\right] + \theta_t^c\left(C_t - \sum_i \pi_i c_{i,t}\right) + \theta_t^n\left(N_t - \sum_i \pi_i e_i n_{i,t}\right)$$

where θ_t^c and θ_t^n are Lagrange multipliers. The first order conditions are

$$[c_{i,t}] : \varphi_i \left(c_{i,t} - \chi\frac{\kappa}{1+\kappa}\left(n_{i,t}\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma} = \theta_t^c, \quad \forall\, t \geq 0,$$

$$[n_{i,t}] : -\varphi_i \left(c_{i,t} - \chi\frac{\kappa}{1+\kappa}\left(n_{i,t}\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma} \chi\left(n_{i,t}\right)^{\frac{1}{\kappa}} = e_i\theta_t^n, \quad \forall\, t \geq 0,$$

rearranging we yield at

$$[c_{i,t}] : \left(c_{i,t} - \chi \frac{\kappa}{1+\kappa} (n_{i,t})^{1+\frac{1}{\kappa}} \right) = (\theta_t^c)^{-\frac{1}{\sigma}} (\varphi_i)^{\frac{1}{\sigma}} \tag{4.11}$$

$$[n_{i,t}] : n_{i,t} = \left(\frac{1}{\chi} \right)^{\kappa} \left(\frac{\theta_t^n}{\theta_t^c} \right)^{\kappa} (e_i)^{\kappa} \tag{4.12}$$

and summing across types (given that $N_t = \sum_i \pi_i e_i n_{i,t}$)

$$C_t - \chi \frac{\kappa}{1+\kappa} \sum_i \pi_i (n_{i,t})^{1+\frac{1}{\kappa}} = (\theta_t^c)^{-\frac{1}{\sigma}} \sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}} \tag{4.13}$$

$$N_t = \left(\frac{1}{\chi} \frac{\theta_t^n}{\theta_t^c} \right)^{\kappa} \sum_i \pi_i (e_i)^{1+\kappa}. \tag{4.14}$$

Now divide (4.12) by (4.14) to get

$$\frac{n_{i,t}}{N_t} = \frac{(e_i)^{\kappa}}{\sum_i \pi_i (e_i)^{1+\kappa}} \Rightarrow n_{i,t} = \frac{(e_i)^{\kappa}}{\sum_i \pi_i (e_i)^{1+\kappa}} N_t$$

Denote

$$\omega_i^n \equiv \frac{(e_i)^{\kappa}}{\sum_i \pi_i (e_i)^{1+\kappa}} \tag{4.15}$$

so that

$$n_{i,t}^m = \omega_i^n N_t. \tag{4.16}$$

Analogously, divide (4.11) by (4.13) to get

$$\frac{\left(c_{i,t} - \chi \frac{\kappa}{1+\kappa} (n_{i,t})^{1+\frac{1}{\kappa}} \right)}{\left(C_t - \chi \frac{\kappa}{1+\kappa} \sum_i \pi_i (n_{i,t})^{1+\frac{1}{\kappa}} \right)} = \frac{(\theta_t^c)^{-\frac{1}{\sigma}} (\varphi_i)^{\frac{1}{\sigma}}}{(\theta_t^c)^{-\frac{1}{\sigma}} \sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}}}$$

which rearranging, using (4.16), leads to

$$c_{i,t} = \frac{(\varphi_i)^{\frac{1}{\sigma}}}{\sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}}} C_t - \chi \frac{\kappa}{1+\kappa} N_t^{1+\frac{1}{\kappa}} \left[\frac{(\varphi_i)^{\frac{1}{\sigma}}}{\sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}}} \sum_i \pi_i (\omega_i^n)^{1+\frac{1}{\kappa}} - (\omega_i^n)^{1+\frac{1}{\kappa}} \right]$$

Denote

$$\omega_i^c \equiv \frac{(\varphi_i)^{\frac{1}{\sigma}}}{\sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}}}, \tag{4.17}$$

and, it follows that,

$$c_{i,t}^m = \omega_i^c C_t - \chi \frac{\kappa}{1+\kappa} N_t^{1+\frac{1}{\kappa}} \left[\omega_i^c \sum_i \pi_i (\omega_i^n)^{1+\frac{1}{\kappa}} - (\omega_i^n)^{1+\frac{1}{\kappa}} \right]. \tag{4.18}$$

Using (4.16) and (4.18) we can write aggregate utility $U(C_t, N_t; \varphi)$ in terms of the aggregates C_t, N_t and the vector of market weights φ as follows

$$U(C_t, N_t; \varphi) = \sum_i \pi_i \varphi_i \left[\frac{1}{1-\sigma} \left(\omega_i^c C_t - \chi \frac{\kappa}{1+\kappa} N_t^{1+\frac{1}{\kappa}} \left[\omega_i^c \sum_i \pi_i (\omega_i^n)^{1+\frac{1}{\kappa}} - (\omega_i^n)^{1+\frac{1}{\kappa}} \right] - \chi \frac{\kappa}{1+\kappa} (\omega_i^n N_t)^{1+\frac{1}{\kappa}} \right)^{1-\sigma} \right]$$
$$= \sum_i \pi_i \varphi_i (\omega_i^c)^{1-\sigma} \left[\frac{1}{1-\sigma} \left(C_t - \left(\sum_i \pi_i (\omega_i^n)^{1+\frac{1}{\kappa}} \right) \chi \frac{\kappa}{1+\kappa} N_t^{1+\frac{1}{\kappa}} \right)^{1-\sigma} \right].$$

Let

$$\Omega^c \equiv \sum_i \pi_i \varphi_i (\omega_i^c)^{1-\sigma} = \sum_i \pi_i \varphi_i \left(\frac{(\varphi_i)^{\frac{1}{\sigma}}}{\sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}}} \right)^{1-\sigma} = \frac{\sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}}}{\left(\sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}} \right)^{1-\sigma}} = \left(\sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}} \right)^{\sigma} \tag{4.19}$$

where the second equality follows from the definition of ω^c given by (4.17). Moreover define

$$\Omega^n \equiv \left(\sum_i \pi_i (\omega_i^n)^{1+\frac{1}{\kappa}} \right) = \left(\sum_i \pi_i (e_i)^{1+\kappa} \right)^{-\frac{1}{\kappa}}$$

where the equality comes from the definition of ω^n given by (4.15). Thus we arrive at a simplified formula,

$$U(C_t, N_t; \varphi) = \Omega^c \frac{\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa} (N_t)^{1+\frac{1}{\kappa}} \right)^{1-\sigma}}{1-\sigma}. \tag{4.20}$$

4.3 Implementability condition

Using the simple characterization from the previous section we can now derive the implementability condition. Applying the envelope theorem to problem (4.10) we get

$$U_C(C_t, N_t; \varphi) = \theta_t^c, \quad \text{and} \quad U_N(C_t, N_t; \varphi) = \theta_t^n.$$

On the other hand, from the first order conditions of the problem (4.10) we have

$$\varphi_i u_c(c_{i,t}, n_{i,t}) = \theta_t^c, \quad \text{and} \quad \frac{\varphi_i u_n(c_{i,t}, n_{i,t})}{e_i} = \theta_t^n.$$

It follows that

$$U_C(C_t, N_t; \varphi) = \varphi_i u_c(c_{i,t}, n_{i,t}), \tag{4.21}$$
$$U_N(C_t, N_t; \varphi) = \frac{\varphi_i u_n(c_{i,t}, n_{i,t})}{e_i}. \tag{4.22}$$

In any competitive equilibrium these optimality conditions must hold for every agent i. Hence, using (4.21) and (4.22), we obtain

$$\frac{U_N\left(C_t,N_t;\varphi\right)}{U_C\left(C_t,N_t;\varphi\right)}=\frac{u_n\left(c_{i,t},n_{i,t}\right)}{u_c\left(c_{i,t},n_{i,t}\right)e_i}=-w_t\frac{\left(1-\tau_t^n\right)}{\left(1+\tau^c\right)} \tag{4.23}$$

and

$$\frac{U_C\left(C_t,N_t;\varphi\right)}{U_C\left(C_0,N_0;\varphi\right)}=\frac{u_c\left(c_{i,t},n_{i,t}\right)}{u_c\left(c_{i,0},n_{i,0}\right)}=\frac{p_t}{\beta^t}. \tag{4.24}$$

Given the relationships above we can derive the implementation condition which relies only on the aggregates C_t, N_t and market weights φ. Let $c_{i,t}^m\left(C_t,N_t;\varphi\right)$ and $n_{i,t}^m\left(C_t,N_t;\varphi\right)$ be the arg max of problem (4.10) given by the (4.18) and (4.16) respectively. The budget constraint of agent i implies

$$\sum_{t=0}^{\infty}p_t\left(c_{i,t}^m\left(C_t,N_t;\varphi\right)-\frac{\left(1-\tau_t^n\right)}{\left(1+\tau^c\right)}w_te_in_{i,t}^m\left(C_t,N_t;\varphi\right)\right)\leq\frac{R_0a_{i,0}+T}{\left(1+\tau^c\right)},$$

which using (4.23) and (4.23) can be restated as

$$\sum_{t=0}^{\infty}\beta^t\left(U_C\left(C_t,N_t;\varphi\right)c_{i,t}^m\left(C_t,N_t;\varphi\right)+U_N\left(C_t,N_t;\varphi\right)e_in_{i,t}^m\left(C_t,N_t;\varphi\right)\right)\leq U_C\left(C_0,N_0;\varphi\right)\left(\frac{R_0a_{i,0}+T}{1+\tau^c}\right),\quad\forall\, i. \tag{4.25}$$

The following Proposition follows immediately from the arguments above.

Proposition 3 *An aggregate allocation $\{C_t,N_t,K_{t+1}\}_{t=0}^{\infty}$ can be supported by a competitive equilibrium if and only if the resource constraints (4.2) hold and there exist market weights φ and a lump-sum tax T so that the implementability conditions (4.25) hold for all $i\in I$. Individual allocations can then be computed using functions $c_{i,t}^m$ and $n_{i,t}^m$, prices and taxes can be computed using the usual equilibrium conditions.*

In the Ramsey problem considered in this paper we restrict the capital income tax to be less than or equal to one. The following Lemma is useful to define our Ramsey problem taking this restriction into consideration.

Lemma 4 *In any competitive equilibrium $\tau_t^k\leq 1$ if and only if $U_C\left(C_t,N_t;\varphi\right)\geq\beta U_C\left(C_{t+1},N_{t+1};\varphi\right)$.*

Proof. $(\Rightarrow)$: Take any competitive equilibrium with $\tau_t^k\leq 1$. Then the first order conditions for agent i imply

$$\begin{aligned}c_{i,t}:&\ \frac{\beta^t}{p_t}u_c\left(c_{i,t},n_{i,t}\right)=\gamma\\ c_{i,t+1}:&\ \frac{\beta^{t+1}}{p_{t+1}}u_c\left(c_{i,t+1},n_{i,t+1}\right)=\gamma\end{aligned}$$

and thus

$$u_c\left(c_{i,t},n_{i,t}\right)=\beta\frac{p_t}{p_{t+1}}u_c\left(c_{i,t+1},n_{i,t+1}\right).$$

Since $R_t \equiv 1 + \left(1 - \tau_t^k\right)(r_t - \delta)$, for $t \geq 0$ and $p_t = R_{t+1} p_{t+1}$ we get

$$u_1^i (c_{i,t}, n_{i,t}) = \beta \left(1 + \left(1 - \tau_t^k\right)(r_t - \delta)\right) u_1^i (c_{i,t+1}, n_{i,t+1}).$$

Further using $U_C (C_t, N_t; \varphi) = \varphi_i u_c (c_{i,t}, n_{i,t})$ from (4.21) we get

$$U_C (C_t, N_t; \varphi) = \beta \left(1 + \left(1 - \tau_t^k\right)(r_t - \delta)\right) U_C (C_{t+1}, N_{t+1}; \varphi)$$

and since $\tau_t \leq 1$ and hence $\left(1 - \tau_t^k\right) > 0$

$$U_C (C_t, N_t; \varphi) = \beta \left(1 + \left(1 - \tau_t^k\right)(r_t - \delta)\right) U_C (C_{t+1}, N_{t+1}; \varphi) \geqslant \beta U_C (C_{t+1}, N_{t+1}; \varphi).$$

which completes the first part of the proof.
$(\Leftarrow)$: Suppose that $U_C (C_t, N_t; \varphi) \geq \beta U_C (C_{t+1}, N_{t+1}; \varphi)$. In any competitive equilibrium we have

$$U_C (C_t, N_t; \varphi) = \beta \left(1 + \left(1 - \tau_t^k\right)(r_t - \delta)\right) U_C (C_{t+1}, N_{t+1}; \varphi)$$

thus by $U_C (C_t, N_t; \varphi) \geq \beta U_C (C_{t+1}, N_{t+1}; \varphi)$

$$\beta \left(1 + \left(1 - \tau_t^k\right)(r_t - \delta)\right) U_C (C_{t+1}, N_{t+1}; \varphi) \geq \beta U_C (C_{t+1}, N_{t+1}; \varphi)$$

implying

$$\left(1 + \left(1 - \tau_t^k\right)(r_t - \delta)\right) \geq 1$$

and hence

$$\tau_t^k \leq 1.$$

■

4.4 Ramsey Problem

Let $\lambda \equiv \{\lambda_i\}$ be the planner's welfare weight on type i, with $\sum_i \pi_i \lambda_i = 1$. Notice that the fact that τ_0^n is given together with the equilibrium condition $(1 - \tau_0^n) w_0 = -U_N (0) / U_C (0)$ is equivalent to

$$N_0 = \bar{N}_0, \tag{4.26}$$

where $\bar{N}_0$ is defined implicitly as a function of variables given to the Ramsey planner,

$$(1 - \tau_0^n) f_N \left(K_0, \bar{N}_0\right) = \Omega^n \chi \left(\bar{N}_0\right)^{\frac{1}{\kappa}}.$$

Hence, the planner cannot choose N_0. Thus, the Ramsey planner problem is

$$\max_{\{C_t, N_{t+1}, K_{t+1}\}_{t=0}^{\infty}, T, \varphi} \sum_{t,i} \lambda_i \pi_i \beta^t u\left(c_{i,t}^m\left(C_t, N_t; \varphi\right), n_{i,t}^m\left(C_t, N_t; \varphi\right)\right)$$

subject to

$$\sum_{t=0}^{\infty} \beta^t \left(U_C\left(C_t, N_t; \varphi\right) c_{i,t}^m\left(C_t, N_t; \varphi\right) + U_N\left(C_t, N_t; \varphi\right) e_i n_{i,t}^m\left(C_t, N_t; \varphi\right)\right) \leq U_C\left(C_0, N_0; \varphi\right)\left(\frac{R_0 a_{i,0} + T}{(1+\tau^c)}\right), \quad \forall\, i,$$

$$C_t + G_t + K_{t+1} = F\left(K_t, N_t\right) + (1-\delta) K_t, \quad \forall\, t \geq 0,$$
$$U_C\left(C_t, N_t; \varphi\right) \geq \beta U_C\left(C_{t+1}, N_{t+1}; \varphi\right), \quad \forall\, t \geq 0.$$

Define

$$\begin{aligned} W\left(C_t, N_t; \varphi, \mu, \lambda\right) \equiv & \sum_i \pi_i \lambda_i u\left(c_{i,t}^m\left(C_t, N_t; \varphi\right), n_{i,t}^m\left(C_t, N_t; \varphi\right)\right) \\ & + \sum_i \pi_i \mu_i \left[U_C\left(C_t, N_t; \varphi\right) c_{i,t}^m\left(C_t, N_t; \varphi\right) + U_N\left(C_t, N_t; \varphi\right) e_i n_{i,t}^m\left(C_t, N_t; \varphi\right)\right] \end{aligned}$$

where μ_i is the Lagrange multiplier on the implementability constraint of agent i, and $\mu \equiv \{\mu_i\}$. Rewrite the Ramsey problem as

$$\max_{\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}, T, \varphi} \sum_{t,i} \beta^t W\left(C_t, N_t; \varphi, \mu, \lambda\right) - U_C\left(C_0, N_0; \varphi\right) \sum_i \pi_i \mu_i \left(\frac{R_0 a_{i,0} + T}{1+\tau^c}\right)$$

subject to

$$C_t + G_t + K_{t+1} = F\left(K_t, N_t\right) + (1-\delta) K_t, \quad \forall\, t \geq 0,$$
$$U_C\left(C_t, N_t; \varphi\right) \geq \beta U_C\left(C_{t+1}, N_{t+1}; \varphi\right), \quad \forall\, t \geq 0,$$

where ν_t and η_t are the Lagrange multipliers on the feasibility and $\tau_t^k \leq 1$ constraint respectively.

4.4.1 From first order conditions to taxes

Define $R_t^* \equiv 1 + r_t$ and

$$\eta_{-1} \equiv \frac{R_0}{\beta\left(1+\tau^c\right)} \sum_i \pi_i \mu_i a_{i,0}.$$

The first order conditions are

$$C_t : W_C\left(C_t, N_t; \varphi, \mu, \lambda\right) - \nu_t + U_{CC}\left(C_t, N_t; \varphi\right)\left(\eta_t - \beta \eta_{t-1}\right) = 0, \quad \forall\, t \geq 0, \tag{4.27}$$

$$N_t : W_N\left(C_t, N_t; \varphi, \mu, \lambda\right) + \nu_t F_N\left(K_t, N_t\right) + U_{CN}\left(C_t, N_t; \varphi\right)\left(\eta_t - \beta\eta_{t-1}\right) = 0, \quad \forall\, t \geq 1, \tag{4.28}$$

$$K_{t+1} : -\nu_t + \left[F_K\left(K_{t+1}, N_{t+1}\right) + (1-\delta)\right]\nu_{t+1} = 0, \quad \forall\, t \geq 0, \tag{4.29}$$

$$T : -U_C\left(C_0, N_0; \varphi\right)\frac{\sum_i \pi_i \mu_i}{1+\tau^c} = 0, \quad \forall\, t \geq 0. \tag{4.30}$$

Dividing (4.28) by (4.27) gives

$$F_N\left(K_t, N_t\right) = -\frac{W_N\left(C_t, N_t; \varphi, \mu, \lambda\right) + U_{CN}\left(C_t, N_t; \varphi\right)\left(\eta_t - \eta_{t-1}\beta\right)}{W_C\left(C_t, N_t; \varphi, \mu, \lambda\right) + U_{CC}\left(C_t, N_t; \varphi\right)\left(\eta_t - \eta_{t-1}\beta\right)} \tag{4.31}$$

and using the intertemporal condition (4.29) we get

$$R^*_{t+1} = \frac{W_C\left(C_t, N_t; \varphi, \mu, \lambda\right) + U_{CC}\left(C_t, N_t; \varphi\right)\left(\eta_t - \eta_{t-1}\beta\right)}{\beta W_C\left(C_{t+1}, N_{t+1}; \varphi, \mu, \lambda\right) + U_{CC}\left(C_{t+1}, N_{t+1}; \varphi\right)\left(\eta_{t+1} - \eta_t\beta\right)} \tag{4.32}$$

These two equations can be used to back out the optimal taxes on labor and capital income.

Plugging (4.31) into (4.23) implies

$$\frac{U_N\left(C_t, N_t; \varphi\right)}{U_C\left(C_t, N_t; \varphi\right)} = \frac{W_N\left(C_t, N_t; \varphi, \mu, \lambda\right) + U_{CN}\left(C_t, N_t; \varphi\right)\left(\eta_t - \eta_{t-1}\beta\right)}{W_C\left(C_t, N_t; \varphi, \mu, \lambda\right) + U_{CC}\left(C_t, N_t; \varphi\right)\left(\eta_t - \eta_{t-1}\beta\right)}\frac{\left(1-\tau^n_t\right)}{\left(1+\tau^c\right)},$$

which can be rearranged into

$$\frac{\tau^n_t + \tau^c}{1+\tau^c} = 1 - \left(\frac{U_N\left(C_t, N_t; \varphi\right)}{U_C\left(C_t, N_t; \varphi\right)}\right)\left(\frac{W_C\left(C_t, N_t; \varphi, \mu, \lambda\right) + U_{CC}\left(C_t, N_t; \varphi\right)\left(\eta_t - \eta_{t-1}\beta\right)}{W_N\left(C_t, N_t; \varphi, \mu, \lambda\right) + U_{CN}\left(C_t, N_t; \varphi\right)\left(\eta_t - \eta_{t-1}\beta\right)}\right). \tag{4.33}$$

In any competitive equilibrium (4.24) holds, which together with $p_t = R_t p_{t+1}$implies

$$\frac{U_C\left(C_{t+1}, N_{t+1}; \varphi\right)}{U_C\left(C_t, N_t; \varphi\right)}\beta R_{t+1} = 1.$$

Substituting this into (4.32) it follows that

$$\frac{R_{t+1}}{R^*_{t+1}} = \frac{W_C\left(C_{t+1}, N_{t+1}; \varphi, \mu, \lambda\right) + U_{CC}\left(C_{t+1}, N_{t+1}; \varphi\right)\left(\eta_{t+1} - \eta_t\beta\right)}{W_C\left(C_t, N_t; \varphi, \mu, \lambda\right) + U_{CC}\left(C_t, N_t; \varphi\right)\left(\eta_t - \eta_{t-1}\beta\right)}\frac{U_C\left(C_t, N_t; \varphi\right)}{U_C\left(C_{t+1}, N_{t+1}; \varphi\right)}. \tag{4.34}$$

4.4.2 Explicit formulas for U and its derivatives

From (4.20) we have

$$U\left(C_t, N_t; \varphi\right) = \sum_i \pi_i \varphi_i \left(\omega^c_i\right)^{1-\sigma}\left[\frac{1}{1-\sigma}\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa} N_t^{1+\frac{1}{\kappa}}\right)^{1-\sigma}\right]$$

$$= \sum_i \pi_i \varphi_i u\left(c^m_{i,t}\left(C_t, N_t; \varphi\right), n^m_{i,t}\left(C_t, N_t; \varphi\right)\right)$$

and using the GHH utility function we can calculate

$$U_C\left(C_t, N_t; \varphi\right) = \Omega^c\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma}$$
$$U_N\left(C_t, N_t; \varphi\right) = -\Omega^c\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma} \Omega^n \chi \left(N_t\right)^{\frac{1}{\kappa}}$$
$$U_{CC}\left(C_t, N_t; \varphi\right) = -\sigma\Omega^c\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma-1}$$
$$U_{CN}\left(C_t, N_t; \varphi\right) = \sigma\Omega^c\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma-1} \Omega^n \chi \left(N_t\right)^{\frac{1}{\kappa}}$$

4.4.3 Explicit formulas for W and its derivatives

It follows from the derivatives of U and equations (4.16) and (4.18) that

$$U_C\left(C_t, N_t; \varphi\right) c^m_{i,t}\left(C_t, N_t; \varphi\right) = \Omega^c\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma} \left\{\omega^c_i C_t - \chi \frac{\kappa}{1+\kappa} N_t^{1+\frac{1}{\kappa}}\left[\omega^c_i \sum_i \pi_i \left(\omega^n_i\right)^{1+\frac{1}{\kappa}} - \left(\omega^n_i\right)^{1+\frac{1}{\kappa}}\right]\right\} \tag{4.35}$$

$$U_N\left(C_t, N_t; \varphi\right) e_i n^m_{i,t}\left(C_t, N_t; \varphi\right) = -\Omega^c\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma} \omega^n_i e_i \Omega^n \chi \left(N_t\right)^{\frac{1+\kappa}{\kappa}}. \tag{4.36}$$

Substituting (4.20), (4.35) and (4.36) into the definition of $W\left(C_t, N_t; \varphi, \mu, \lambda\right)$ we get

$$\begin{aligned} W\left(C_t, N_t; \varphi, \mu, \lambda\right) = & \sum_i \pi_i \lambda_i \left(\omega^c_i\right)^{1-\sigma}\left[\frac{1}{1-\sigma}\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa} N_t^{1+\frac{1}{\kappa}}\right)^{1-\sigma}\right] \\ & + \sum_i \pi_i \mu_i \Omega^c \left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma}\left\{\omega^c_i C_t - \chi \frac{\kappa}{1+\kappa} N_t^{1+\frac{1}{\kappa}}\left[\omega^c_i \sum_i \pi_i \left(\omega^n_i\right)^{1+\frac{1}{\kappa}} - \left(\omega^n_i\right)^{1+\frac{1}{\kappa}}\right]\right\} \\ & - \sum_i \pi_i \mu_i \Omega^c \left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma} e_i \omega^n_i \Omega^n \chi \left(N_t\right)^{\frac{1+\kappa}{\kappa}} \end{aligned} \tag{4.37}$$

Collecting terms and simplifying we obtain

$$\begin{aligned} W\left(C_t, N_t; \varphi, \mu, \lambda\right) = & \frac{1}{1-\sigma}\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma}\left\{C_t\left(\sum_i \pi_i \left[\lambda_i \left(\omega^c_i\right)^{1-\sigma} + \left(1-\sigma\right)\mu_i \Omega^c \omega^c_i\right]\right) - \chi \frac{\kappa}{1+\kappa}\Omega^n \left(N_t\right)^{\frac{1+\kappa}{\kappa}}\right. \\ & \left. * \left\{\sum_i \pi_i \left[\lambda_i \left(\omega^c_i\right)^{1-\sigma} + \left(1-\sigma\right)\mu_i \Omega^c \omega^c_i\right] + \left(1-\sigma\right)\Omega^c \sum_i \pi_i \mu_i \left[\frac{1+\kappa}{\kappa} e_i \omega^n_i - \frac{\left(\omega^n_i\right)^{1+\frac{1}{\kappa}}}{\Omega^n}\right]\right\}\right\}. \end{aligned}$$

Now define the following constants

$$\Phi \equiv \sum_i \pi_i \left(\lambda_i \left(\omega_i^c\right)^{1-\sigma} + (1-\sigma)\,\mu_i \omega_i^c \Omega^c\right) = \left(\Omega^c\right)^{\frac{\sigma-1}{\sigma}} \sum_i \pi_i (\varphi_i)^{\frac{1}{\sigma}} \left(\frac{\lambda_i}{\varphi_i} + (1-\sigma)\,\mu_i\right), \tag{4.38}$$

$$\Psi \equiv \Omega^c \sum_i \pi_i \mu_i \left(\frac{1+\kappa}{\kappa} e_i \omega_i^n - \frac{\left(\omega_i^n\right)^{\frac{1+\kappa}{\kappa}}}{\Omega^n}\right) = \frac{\Omega^c}{\kappa} \sum_i \pi_i \mu_i e_i \omega_i^n. \tag{4.39}$$

Using these constants we can then write

$$W\left(C_t, N_t; \varphi, \mu, \lambda\right) = \frac{1}{1-\sigma}\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma}\left(\Phi C_t - \left(\Phi + (1-\sigma)\,\Psi\right)\Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{\frac{1+\kappa}{\kappa}}\right) \tag{4.40}$$

Taking derivatives we obtain

$$W_C\left(C_t, N_t; \varphi, \mu, \lambda\right) = \frac{1}{1-\sigma}\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma}\left[\Phi - \sigma \frac{\left(\Phi C_t - \left(\Phi + (1-\sigma)\,\Psi\right)\Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{\frac{1+\kappa}{\kappa}}\right)}{\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)}\right], \tag{4.41}$$

$$\begin{aligned} W_N\left(C_t, N_t; \varphi, \mu, \lambda\right) = {} & \frac{1}{1-\sigma}\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)^{-\sigma} \\ & * \left[\Omega^n \chi \left(N_t\right)^{\frac{1}{\kappa}}\left(\frac{\sigma\left(\Phi C_t - \left(\Phi + (1-\sigma)\,\Psi\right)\Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{\frac{1+\kappa}{\kappa}}\right)}{\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right)} - \left(\Phi + (1-\sigma)\,\Psi\right)\right)\right]. \end{aligned} \tag{4.42}$$

4.4.4 Substituting the derivatives into the tax formulas

Substituting the derivatives into equation (4.33) we get

$$\frac{\tau_t^n + \tau^c}{1+\tau^c} = 1 + \frac{-\frac{\sigma\left(\Phi C_t - (\Phi + (1-\sigma)\Psi)\Omega^n \chi \frac{\kappa}{1+\kappa}(N_t)^{\frac{1+\kappa}{\kappa}}\right)}{\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}(N_t)^{1+\frac{1}{\kappa}}\right)} + \Phi - \frac{\sigma \Omega^c (\eta_t - \eta_{t-1}\beta)}{\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}(N_t)^{1+\frac{1}{\kappa}}\right)}}{\frac{\sigma\left(\Phi C_t - (\Phi + (1-\sigma)\Psi)\chi \frac{\kappa}{1+\kappa}\Omega^n (N_t)^{\frac{1+\kappa}{\kappa}}\right)}{\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}(N_t)^{1+\frac{1}{\kappa}}\right)} - \left(\Phi + (1-\sigma)\,\Psi\right) + \frac{\sigma \Omega^c (\eta_t - \eta_{t-1,}\beta)}{\left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa}(N_t)^{1+\frac{1}{\kappa}}\right)}}, \tag{4.43}$$

and simplifying we obtain

$$\frac{\tau_t^n + \tau^c}{1+\tau^c} = \frac{\Psi \Theta_t}{\Phi \Theta_t + \Psi\left(\Theta_t + \sigma\right) + \sigma \Upsilon_t\left(\beta \eta_{t-1} - \eta_t\right)}. \tag{4.44}$$

where

$$\Theta_t \equiv \frac{C_t}{\Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}} - 1 \tag{4.45}$$

$$\Upsilon_t \equiv \frac{1}{(1-\sigma)} \frac{\Omega^c}{\Omega^n \chi \frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}} \tag{4.46}$$

Analogously, substituting the derivatives into (4.34) and simplifying leads to

$$\frac{R_{t+1}}{R^*_{t+1}} = \frac{\Phi\Theta_{t+1} + \sigma\Psi + \sigma\Upsilon_{t+1}\left(\beta\eta_t - \eta_{t+1}\right)}{\Phi\Theta_t + \sigma\Psi + \sigma\Upsilon_t\left(\beta\eta_{t-1} - \eta_t\right)} \frac{\Theta_t}{\Theta_{t+1}} \tag{4.47}$$

4.5 Detailed proof of Proposition for Economy 2

Lemma 5 *If $e_i = 1$ for all $i \in I$, then $\Psi = 0$ and $\Phi > 0$.*

Proof. If $e_i = 1$ for all $i \in I$, then it follows from the definition of Ψ that

$$\Psi = \frac{\Omega^c}{\kappa} \frac{\sum_j \pi_j \mu_j \left(e_j\right)^{1+\kappa}}{\sum_j \pi_j \left(e_j\right)^{1+\kappa}} = \frac{\Omega^c}{\kappa} \frac{\sum_j \pi_j \mu_j}{\sum_j \pi_j} = 0$$

where the last equality follows from equation (4.30). Next, notice that

$$u\left(c^m_{i,t}\left(C_t, N_t; \varphi\right), n^m_{i,t}\left(C_t, N_t; \varphi\right)\right) = \frac{\left(\omega^c_i\right)^{1-\sigma}}{1-\sigma} \left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa} \left(N_t\right)^{\frac{1+\kappa}{\kappa}}\right)^{1-\sigma}$$

and, therefore, the solution to the problem must satisfy $C_t > \Omega^n \chi \frac{\kappa}{1+\kappa} \left(N_t\right)^{\frac{1+\kappa}{\kappa}}$ for any finite $t \geq 0$. Otherwise, the objective function of the Ramsey problem would be $-\infty$. On the other hand, since $\Psi = 0$, it follows from equation (4.40) that

$$W\left(C_t, N_t; \varphi, \mu, \lambda\right) = \frac{\Phi}{1-\sigma} \left(C_t - \Omega^n \chi \frac{\kappa}{1+\kappa} \left(N_t\right)^{\frac{1+\kappa}{\kappa}}\right)^{1-\sigma}.$$

Fix some finite t, if $\Phi \leq 0$ then reducing C_t to 0 is feasible and would weakly increase welfare which is a contradiction. ■

Proposition 4 *There exists a finite $t^* \geq 1$ such that the optimal tax system is given by $\tau^k_t = 1$ for $1 \leq t \leq t^*$ and $\tau^k_t = 0$ for all $t > t^*$; and $\tau^n_t = -\tau^c$ for all $t \geq 1$.*

Proof. From Lemma 5 and from the formula pinning down the labor tax (4.44) we have

$$\frac{\tau^n_t + \tau^c}{1 + \tau^c} = 0 \;\Rightarrow\; \tau^n_t = -\tau^c, \quad t \geq 1.$$

Next suppose $\eta_t = 0$ for all $t \geq 0$. Evaluating (4.47) for period 0 we get

$$\frac{R_1}{R^*_1} = \frac{\Phi\Theta_1 + \sigma\Psi + \sigma\Upsilon_1\left(\eta_0\beta - \eta_1\right)}{\Phi\Theta_0 + \sigma\Psi + \sigma\Upsilon_0\left(\eta_{-1}\beta - \eta_0\right)} \frac{\Theta_0}{\Theta_1} = \frac{\Phi\Theta_1 + \sigma\Psi}{\Phi\Theta_0 + \sigma\Psi + \sigma\Upsilon_0\eta_{-1}\beta}$$

which, since $\Psi = 0$, implies that

$$\frac{\Phi\Theta_0 R_1^*}{\beta\sigma\Upsilon_0 R_1} = \frac{\Phi\Theta_0}{\beta\sigma\Upsilon_0} + \eta_{-1}$$

It follows that, if

$$\eta_{-1} < \frac{\Phi\Theta_0\left(R_1^* - 1\right)}{\beta\sigma\Upsilon_0}$$

then

$$\frac{\Phi\Theta_0 R_1^*}{\beta\sigma\Upsilon_0 R_1} < \frac{\Phi\Theta_0}{\beta\sigma\Upsilon_0 R_1} \;\Rightarrow\; R_1 > 1 \;\Rightarrow\; \tau_1^k < 1.$$

Moreover, if

$$\eta_{-1} > -\frac{\Phi\Theta_0}{\beta\sigma\Upsilon_0}$$

then

$$\frac{\Phi\Theta_0 R_1^*}{\beta\sigma\Upsilon_0 R_1} > 0 \;\Rightarrow\; R_1 \text{ is finite} \;\Rightarrow\; \tau_1^k \text{ is finite.}$$

Hence, in particular,

$$-\frac{1}{\beta}\frac{\Phi\Theta_0}{\Upsilon_0\sigma} \equiv P_1 < \eta_{-1} < M_1 \equiv \frac{1}{\beta}\frac{\Phi\Theta_0\left(R_1^* - 1\right)}{\Upsilon_0\sigma} \;\Rightarrow\; \tau_1^k < 1 \text{ and } \tau_1^k \text{ is finite.}$$

For $t \geq 1$ the fact that $\eta_t = 0$ for all $t \geq 0$ and $\Psi = 0$ imply in (4.47) that

$$\frac{R_{t+1}}{R_{t+1}^*} = \frac{\Phi\Theta_{t+1}}{\Phi\Theta_t}\frac{\Theta_t}{\Theta_{t+1}} = 1$$

hence $\tau_t^k = 0$ for all $t \geq 2$. Thus, if $P_1 < \eta_{-1} < M_1$ then the upper bound constraint on the τ_t^k is never binding and, in fact, $\eta_t = 0$ for all $t \geq 0$.

Now, pick some finite t^* and suppose $\eta_t > 0$ for $t \leq t^* - 2$ and $\eta_t = 0$ for all $t \geq t^* - 1$. Then, evaluating (4.47) at $t = t^* - 1$ gives

$$\frac{R_{t^*}}{R_{t^*}^*} = \frac{\Phi\Theta_{t^*-1}}{\Phi\Theta_{t^*-1} + \Upsilon_{t^*-1}\sigma\beta\eta_{t^*-2}}$$

which implies

$$\frac{\Phi\Theta_{t^*-1} R_{t^*}^*}{\beta\sigma\Upsilon_{t^*-1} R_{t^*}} = \frac{\Phi\Theta_{t^*-1}}{\beta\sigma\Upsilon_0} + \eta_{t^*-1}$$

and, analogously to above it follows that

$$-\frac{1}{\beta}\frac{\Phi\Theta_{t^*-1}}{\Upsilon_{t^*-1}\sigma} < \eta_{t^*-2} < \frac{1}{\beta}\frac{\Phi\Theta_{t^*-1}\left(R_t^*-1\right)}{\Upsilon_{t^*-1}\sigma} \;\Rightarrow\; \tau_{t^*}^k < 1 \text{ and } \tau_{t^*}^k \text{ is finite.}$$

For $t \leq t^* - 2$, (4.47) evaluated with $\tau_t^k = 1$ yields

$$\frac{\Phi\Theta_{t+1} + \Upsilon_{t+1}\sigma\left(\beta\eta_t - \eta_{t+1}\right)}{\Phi\Theta_t + \Upsilon_t\sigma\left(\beta\eta_{t-1} - \eta_t\right)}\frac{\Theta_t}{\Theta_{t+1}} = \frac{1}{R_{t+1}^*}, \quad 0 \leq t \leq t^* - 2.$$

Let

$$X_t \equiv \beta\eta_{t-1} - \eta_t,$$

then

$$X_t = \frac{\Phi\Theta_t\left(R_{t+1}^* - 1\right)}{\Upsilon_t\sigma} + \frac{R_{t+1}^*\Upsilon_{t+1}\Theta_t}{\Upsilon_t\Theta_{t+1}}X_{t+1}, \quad 0 \leq t \leq t^* - 2.$$

which is a first-order difference equation on X_t with initial and terminal conditions given by

$$X_0 = \beta\eta_{-1} - \eta_0, \quad \text{and} \quad X_{t^*-1} = \beta\eta_{t^*-2}.$$

Denote

$$A_t \equiv \frac{\left(R_{t+1}^* - 1\right)\Phi\Theta_t}{\Upsilon_t\sigma}, \quad \text{and} \quad B_t \equiv \frac{R_{t+1}^*\Theta_t\Upsilon_{t+1}}{\Theta_{t+1}\Upsilon_t}$$

so that we have the following dynamic system

$$\begin{aligned} X_t &= A_t + B_t X_{t+1}, \quad 0 \leq t \leq t^* - 2 \\ \eta_{t-1} &= \frac{1}{\beta}\left(\eta_t + X_t\right), \quad 0 \leq t \leq t^* - 1 \end{aligned}$$

where the second equation comes directly out of the definition of X_t. In what follows the idea is to transform the bounds imposed on η_{t^*-2} on bounds on η_{-1}. Start with η_{-1}

$$\eta_{-1} = \frac{1}{\beta}\left(\eta_0 + X_0\right)$$

and plug in iteratively for X_t and η_t for all $t \leq t^* - 2$ to get

$$\eta_{-1} = \frac{1}{\beta}\left(\eta_t + X_t\right) = \frac{1}{\beta^{t^*}}\eta_{t^*-1} + \sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}X_{\tau-1}$$

and note that

$$\begin{aligned}
\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}X_{\tau-1} &= \frac{1}{\beta}X_0+\frac{1}{\beta^2}X_1+\frac{1}{\beta^3}X_2+...\\
&= \frac{1}{\beta}\left(A_0+B_0X_1\right)+\frac{1}{\beta^2}\left(A_1+B_1X_2\right)+\frac{1}{\beta^3}X_2+...\\
&= \frac{1}{\beta}\left(A_0+B_0\left(A_1+B_1\left(A_2+B_2X_3\right)\right)\right)+\frac{1}{\beta^2}\left(A_1+B_1\left(A_2+B_2X_3\right)\right)+\frac{1}{\beta^3}\left(A_2+B_2X_3\right)+...\\
&= \frac{1}{\beta}A_0+\frac{1}{\beta^2}A_1+\frac{1}{\beta^3}A_2+\frac{1}{\beta}B_0A_1+\frac{1}{\beta^2}B_1A_2+\frac{1}{\beta}B_0B_1A_2+\frac{1}{\beta^2}B_1B_2X_3+\frac{1}{\beta^3}B_2X_3+\frac{1}{\beta}B_0B_1B_2X_3+...\\
&= \frac{1}{\beta}\left[A_0+B_0A_1+B_0B_1A_2+B_0B_1B_2X_3\right]+\frac{1}{\beta^2}\left[A_1+B_1A_2+B_1B_2X_3\right]+\frac{1}{\beta^3}\left[A_2+B_2X_3\right]+...\\
&= \sum_{s=\tau-1}^{t^*-2}\left(\prod_{j=\tau-1}^{s-1}B_j\right)A_s+\left(\prod_{j=\tau-1}^{t^*-2}B_j\right)X_{t^*-1}
\end{aligned}$$

Hence we get

$$\eta_{-1}=\frac{1}{\beta^{t^*}}\eta_{t^*-1}+\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\left(\sum_{s=\tau-1}^{t^*-2}\left(\prod_{j=\tau-1}^{s-1}B_j\right)A_s+\left(\prod_{j=\tau-1}^{t^*-2}B_j\right)X_{t^*-1}\right). \tag{4.48}$$

Now using the definitions of A_t and B_t, the terminal condition, and the fact that $\eta_{t^*-1}=0$ we get

$$\eta_{-1}=\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\left(\sum_{s=\tau-1}^{t^*-2}\left(\prod_{j=\tau-1}^{s-1}\frac{R^*_{j+1}\Theta_j\Upsilon_{j+1}}{\Theta_{j+1}\Upsilon_j}\right)\frac{\left(R^*_{s+1}-1\right)\Phi\Theta_s}{\Upsilon_s\sigma}+\left(\prod_{j=\tau-1}^{t^*-2}\frac{R^*_{j+1}\Theta_j\Upsilon_{j+1}}{\Theta_{j+1}\Upsilon_j}\right)\beta\eta_{t^*-2}\right)$$

which relates the η_{-1} and η_{t^*-2}. Solving for η_{t^*-2} we get

$$\eta_{t^*-2}=\frac{\eta_{-1}-\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\left(\sum_{s=\tau-1}^{t^*-2}\left(\prod_{j=\tau-1}^{s-1}\frac{R^*_{j+1}\Theta_j\Upsilon_{j+1}}{\Theta_{j+1}\Upsilon_j}\right)\frac{\left(R^*_{s+1}-1\right)\Phi\Theta_s}{\Upsilon_s\sigma}\right)}{\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\left(\prod_{j=\tau-1}^{t^*-2}\frac{R^*_{j+1}\Theta_j\Upsilon_{j+1}}{\Theta_{j+1}\Upsilon_j}\right)\beta}$$

The upper bound on η_{t^*-2},

$$\eta_{t^*-2}<\frac{1}{\beta}\frac{\Phi\Theta_{t^*-1}\left(R^*_t-1\right)}{\Upsilon_{t^*-1}\sigma},$$

is equivalent to

$$\begin{aligned}
\eta_{-1} &< \frac{\Phi}{\sigma}\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\frac{\Theta_{\tau-1}}{\Upsilon_{\tau-1}}\left(\sum_{s=\tau-1}^{t^*-2}\left(\prod_{j=\tau-1}^{s-1}R^*_{j+1}\right)\left(R^*_{s+1}-1\right)\right)+\frac{\Phi}{\sigma}\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\frac{\Theta_{\tau-1}}{\Upsilon_{\tau-1}}\left(\prod_{j=\tau-1}^{t^*-2}R^*_{j+1}\right)\left(R^*_{t^*}-1\right)\\
&= \sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\frac{\left(\prod_{j=\tau}^{t^*}R^*_j-1\right)\Phi\Theta_{\tau-1}}{\Upsilon_{\tau-1}\sigma}
\end{aligned}$$

and the lower bound

$$\eta_{t^*-2} > -\frac{1}{\beta}\frac{\Phi\Theta_{t^*-1}}{\Upsilon_{t^*-1}\sigma}$$

is equivalent to

$$\eta_{-1} > \sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\left(\sum_{s=\tau-1}^{t^*-2}\left(\prod_{j=\tau-1}^{s-1}\frac{R^*_{j+1}\Theta_j\Upsilon_{j+1}}{\Theta_{j+1}\Upsilon_j}\right)\frac{\left(R^*_{s+1}-1\right)\Phi\Theta_s}{\Upsilon_s\sigma}\right) - \frac{1}{\beta}\frac{\Phi\Theta_{t^*-1}}{\Upsilon_{t^*-1}\sigma}\left(\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\left(\prod_{j=\tau-1}^{t^*-2}\frac{R^*_{j+1}\Theta_j\Upsilon_{j+1}}{\Theta_{j+1}\Upsilon_j}\right)\beta\right)$$
$$= -\frac{\Phi}{\sigma}\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\frac{\Theta_{\tau-1}}{\Upsilon_{\tau-1}}.$$

Therefore we obtain that

$$-\sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\frac{\Phi\Theta_{\tau-1}}{\Upsilon_{\tau-1}\sigma} \equiv P_{t^*} < \eta_{-1} < M_{t^*} \equiv \sum_{\tau=1}^{t^*}\frac{1}{\beta^\tau}\frac{\left(\prod_{t=\tau}^{t^*}R^*_t - 1\right)\Phi\Theta_{\tau-1}}{\Upsilon_{\tau-1}\sigma} \Rightarrow \tau^k_{t^*} < 1 \text{ and } \tau^k_{t^*} \text{ is finite.} \quad (4.49)$$

For $t \geq t^*$ the fact that $\eta_t = 0$ for all $t \geq t^* - 1$ and $\Psi = 0$ imply $\tau^k_t = 0$ for all $t \geq t^* + 1$. Thus, if $P_{t^*} < \eta_{-1} < M_{t^*}$, then the upper bound constraint on the τ^k_t is binding only for $t \leq t^* - 1$ and, in fact, $\eta_t > 0$ for $t \leq t^* - 2$ and $\eta_t = 0$ for all $t \geq t^* - 1$.

Finally, notice that

$$\frac{\Theta_t}{\Upsilon_t} = \frac{(\sigma-1)}{\Omega^c}\left(C_t - \Omega^n\chi\frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}\right),$$

and the term $C_t - \Omega^n\chi\frac{\kappa}{1+\kappa}\left(N_t\right)^{1+\frac{1}{\kappa}}$ cannot go to zero faster than β^t, otherwise the objective function of the planner would go to $-\infty$. Therefore,

$$\lim_{t\longrightarrow\infty} P_t = -\infty, \quad \text{and} \quad \lim_{t\longrightarrow\infty} M_t = \infty$$

and since the η_{-1} is finite then the result follows. ■

4.6 Detailed proof of Proposition for Economy 3

Proposition 5 *Assuming capital taxes are bounded only by the positivity of gross interest rates, the optimal labor tax, τ^n_t, can be written as a function of Θ_t given by*

$$\tau^n_t\left(\Theta_t\right) = \frac{\left(1+\tau^c\right)\Psi\Theta_t}{\Phi\Theta_t + \Psi\left(\sigma+\Theta_t\right)} - \tau^c, \quad \textit{for } t \geq 1, \quad (4.50)$$

with sensitivity

$$\Theta_t\frac{d\tau^n_t\left(\Theta_t\right)}{d\Theta_t} = \frac{\sigma\left(\tau^n_t\left(\Theta_t\right)+\tau^c\right)^2}{\left(1+\tau^c\right)\Theta_t}. \quad (4.51)$$

It is optimal to set the capital-income tax rate according to

$$\frac{R_{t+1}}{R^*_{t+1}} = \frac{\tau^n_t + \tau^c}{\tau^n_{t+1} + \tau^c} \frac{1 - \tau^n_{t+1}}{1 - \tau^n_t}, \quad \textit{for } t \geq 1.$$

Proof. In this economy there is no heterogeneity in initial levels of assets, i.e. $a_{i,0} = a_0$ for all $i \in I$. It follows that

$$\eta_{-1} = \frac{R_0}{\beta(1+\tau^c)} \sum_i \pi_i \mu_i a_{i,0} = \frac{R_0}{\beta(1+\tau^c)} a_0 \sum_i \pi_i \mu_i = 0$$

where the last equality follows from (4.30). Since, we do not impose an upper bound on the capital income tax the constraint $U_C(C_t, N_t; \varphi) \geq \beta U_C(C_{t+1}, N_{t+1}; \varphi)$ is dropped and (4.44) becomes

$$\frac{\tau^n_t + \tau^c}{1+\tau^c} = \frac{\Psi\Theta_t}{\Phi\Theta_t + \Psi(\Theta_t + \sigma)} \tag{4.52}$$

Rearranging we get (4.50). The sensitivity of the labor income tax is derived by differentiating the formula above with respect to Θ_t, i.e.

$$\begin{aligned}\frac{d\tau^n_t(\Theta_t)}{d\Theta_t} &= \frac{(1+\tau^c)\Psi(\Phi\Theta_t + \Psi(\sigma+\Theta_t)) - (1+\tau^c)\Psi\Theta_t(\Phi+\Psi)}{(\Phi\Theta_t + \Psi(\sigma+\Theta_t))^2} \\ &= \frac{\sigma}{(1+\tau^c)\Theta_t^2}(\tau^n_t(\Theta_t) + \tau^c)^2\end{aligned}$$

which implies (4.51). In the absence of the upper bound on the capital income tax the (4.47) becomes

$$\frac{R_{t+1}}{R^*_{t+1}} = \frac{\Phi\Theta_{t+1} + \sigma\Psi}{\Phi\Theta_t + \sigma\Psi} \frac{\Theta_t}{\Theta_{t+1}} = \frac{\Psi\Theta_t}{\Phi\Theta_t + \sigma\Psi} \frac{\Phi\Theta_{t+1} + \sigma\Psi}{\Psi\Theta_{t+1}}. \tag{4.53}$$

It follows that

$$\begin{aligned}\frac{R_{t+1}}{R^*_{t+1}} &= \left(\frac{\Psi\Theta_t}{\Phi\Theta_t + \Psi(\Theta_t+\sigma)}\right)\left(\frac{\Phi\Theta_t + \Psi(\sigma+\Theta_t)}{\Phi\Theta_t + \Psi\sigma}\right)\left(\frac{\Phi\Theta_{t+1} + \sigma\Psi}{\Psi\Theta_{t+1}}\right) \\ &= \left(\frac{\tau^n_t + \tau^c}{1+\tau^c}\right)\left(\frac{1+\tau^c}{1-\tau^n_t}\right)\left(\frac{1-\tau^n_{t+1}}{\tau^n_{t+1}+\tau^c}\right) \\ &= \left(\frac{\tau^n_t + \tau^c}{1-\tau^n_t}\right)\left(\frac{1-\tau^n_{t+1}}{\tau^n_{t+1}+\tau^c}\right)\end{aligned}$$

which completes the proof. ■

4.7 Detailed proof of Proposition for Economy 4

Proposition 6 *There exists a finite integer $t^* \geq 1$ such that the optimal tax system is given by $\tau_t^k = 1$ for $1 \leq t < t^*$, τ_t^k follows equation*

$$\frac{R_{t+1}}{R_{t+1}^*} = \frac{\tau_t^n + \tau^c}{1 - \tau_t^n} \frac{1 - \tau_{t+1}^n}{\tau_{t+1}^n + \tau^c} \tag{4.54}$$

for $t > t^*$, τ_t^n evolves according to equation (4.54) for $1 \leq t < t^*$, and τ_t^n is determined by

$$\tau_t^n(\Theta_t) = \frac{(1 + \tau^c)\,\Psi\Theta_t}{\Phi\Theta_t + \Psi(\sigma + \Theta_t)} - \tau^c \tag{4.55}$$

for all $t \geq t^*$.

Proof. The existence of t^* such that $\eta_t > 0$, for $t < t^* - 1$ and $\eta_t = 0$, for all $t \geq t^* - 1$, can be shown using an argument analogous to the one used in the proof of Proposition 4, the only difference being that Ψ is no longer equal to 0. The corresponding P_{t^*} and M_{t^*} are given by

$$\begin{aligned} P_{t^*} &\equiv -\sum_{\tau=1}^{t^*} \frac{1}{\beta^\tau} \frac{\Phi\Theta_{\tau-1} + \Psi\sigma}{\Upsilon_{\tau-1}\sigma} \\ M_{t^*} &\equiv \sum_{\tau=1}^{t^*} \frac{1}{\beta^\tau} \frac{\left(\prod_{t=\tau}^{t^*} R_t^* - 1\right)\Phi\Theta_{\tau-1} + \left(\frac{\Theta_{\tau-1}}{\Theta_{t^*}}\prod_{t=\tau}^{t^*} R_t^* - 1\right)\Psi\sigma}{\Upsilon_{\tau-1}\sigma}. \end{aligned}$$

Since $\eta_t = 0$, for all $t \geq t^* - 1$, the same argument used in Proposition 5 implies that equations (4.54) and (4.55) must be satisfied, for $t \geq t^* - 1$. To show that equation (4.54) is satisfied also for $t \leq t^* - 1$, first solve for $\sigma\Upsilon_t(\eta_t - \beta\eta_{t-1})$ from equation (4.44),

$$\sigma\Upsilon_t(\eta_t - \beta\eta_{t-1}) = \frac{1 - \tau_t^n}{\tau_t^n + \tau^c}\Psi\Theta_t - \Phi\Theta_t - \sigma\Psi$$

and subtitute it into equation (4.47). Finally, notice that since $\tau_t^k = 1$ for $1 \leq t < t^*$ and given τ_0^n , equation (4.54) fully determines the evolution of τ_t^n up to t^*. For the latter periods, equation (4.55) determines τ_t^n and, given then (4.54) determines τ_t^k. ■

4.8 Relationship to Straub and Werning (2014)

The reason why capital taxes should converge to zero, that is the reason why the point made by Straub and Werning (2014) is not applicable to our environment is because unrestricted lump-sum transfers are an available instrument for the planner. When lump-sum transfers are not available the planner might choose to tax capital income because it needs to obtain revenue and it is less distortive than other instruments. When it is available, since lump-sum taxes are not distortive, capital taxes are only chosen by the planner in order to provide redistribution; lump-sum taxes are always a more efficient alternative to obtain revenue.

To see this more clearly, consider environment above without heterogeneity (for simplicity) and without lump-sum transfers. Then the first order condition of the Ramsey problem with respect to C_t, i.e. the analogue of (4.27), is

$$\nu_t = (1-(1-\sigma)\mu)U_C\left(C_t,N_t\right) + \frac{\mu}{1-\sigma}U_{CN}\left(C_t,N_t\right) + \left(\eta_t - \beta\eta_{t-1}\right)U_{CC}\left(C_t,N_t\right), \quad \forall\, t \geq 0,$$

where $\nu_t \geq 0$ is the multiplier on the resource constraint at time t, μ is the multiplier on the incentive-compatibility constraint, and η_t is the multiplier on the upper bound of τ_t^k. The argument in Straub and Werning (2014) can be summarized in this case as follows. Suppose that $\sigma > 1$ and $\eta_t = 0,\ \forall\, t \geq t^*$, then it is possible to choose μ high enough so that $\nu_t < 0$ which is absurd. It follows, therefore, that for μ high enough the upper bound on τ_t^k is always binding. To increase μ one needs to increase the planner's need of revenue, for instance by increasing the amount of government expenditures or of initial debt. This leads to more distortionary taxation, and, if extreme enough, positive capital taxes forever.

When lump-sum taxes are available it is easy to see that the analogue equation to (4.30) implies $\mu = 0$ and in this case $\eta_t = 0,\ \forall\ t \geq t^*$ implies $\nu_t = U_C\left(C_t,N_t\right) \geq 0$. Distortive taxes are no longer used for the sole purpose of obtaining revenue (so $\mu = 0$), but only if it allows the planner to provide redistribution more efficiently than the non-distortive lump-sum instrument.

5 Long-run Optimality Conditions

Here we follow Acikgoz (2015) and Hagedorn et al. (2016) and obtain analogous results to theirs for the GHH utility function. We also describe the algorithm we used to compute the optimal fiscal policy in the long-run using their method. At the end we present the results we obtained from following this procedure.

5.1 Environment

There is a measure one of households. Denote the household's history by $h^t = \{h^{t-1}, e_t\}$ with $h^0 = \{a_0, e_0\}$. Given a sequence of prices and taxes the household solves

$$V(a_0, e_0) = \max_{\{c_t(h^t), n_t(h^t), a_{t+1}(h^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) u\left(c_t\left(h^t\right), n_t\left(h^t\right)\right)$$

subject to

$$(1+\tau^c) c_t\left(h^t\right) + a_{t+1}\left(h^t\right) = \bar{w}_t e_t\left(h^t\right) n_t\left(h^t\right) + (1+\bar{r}_t) a_t\left(h^{t-1}\right) + T_t$$

$$a_{t+1}\left(h^t\right) \geq \underline{a}.$$

where

$$\bar{w}_t \equiv (1-\tau_t^n) w_t \quad \text{and} \quad \bar{r}_t \equiv \left(1-\tau_t^k\right) r_t$$

Given prices, in each period, the representative firm solves

$$\max_{K_t, N_t} F(K_t, N_t) - w_t N_t - r_t K_t.$$

Government finances an exogenous stream of expenditure and lump-sum transfers with taxes on labor and capital or debt

$$G_t + T_t + r_t B_t = B_{t+1} - B_t + \tau^c C_t + \tau_t^n w_t N_t + \tau_t^k r_t (K_t + B_t).$$

5.1.1 Equilibrium

Definition 3 *Given K_0, B_0, an initial distribution $\Pi(h^0)$ and a policy $\pi \equiv \{\tau_t^k, \tau_t^n, T_t\}_{t=0}^{\infty}$, a **competitive equilibrium** is an allocation $\{c_t(h^t), n_t(h^t), a_{t+1}(h^t)\}_{h^t}, K_t, N_t, B_t\}_{t=0}^{\infty}$, a price system $P \equiv \{r_t, w_t\}_{t=0}^{\infty}$, such that for all t:*

1. *Given P and π, $\{c_t(h^t), n_t(h^t), a_{t+1}(h^t)\}$ solve the household's problem;*

2. *Factor prices are set competitively:* $r_t = F_K(K_t, N_t),\ w_t = F_N(K_t, N_t)$*;*

3. *Government budget constraint holds and debt is bounded;*

4. *Markets clear,*

$$N_t = \sum_{h^t} \Pi\left(h^t\right) e_t n_t\left(h^t\right)$$
$$K_t + B_t = \sum_{h^{t-1}} \Pi\left(h^{t-1}\right) a_t\left(h^{t-1}\right).$$

5.1.2 Characterization

First order conditions of the household's problem:

$$\begin{aligned}
\tilde{c}_t\left(h^t\right) &= \frac{\bar{w}_t e_t\left(h^t\right) n_t\left(h^t\right) + \left(1+\bar{r}_t\right) a_t\left(h^{t-1}\right) + T_t - a_{t+1}\left(h^t\right)}{1+\tau^c}, \\
\tilde{n}_t\left(h^t\right) &= \left(\frac{\bar{w}_t e_t\left(h^t\right)}{\left(1+\tau^c\right)\chi}\right)^{\varepsilon}, \\
\tilde{u}_c\left(h^t\right) &\geq \beta\left(1+\bar{r}_{t+1}\right) \sum_{h^{t+1}} \Pi\left(h^{t+1} \mid h^t\right) \tilde{u}_c\left(h^{t+1}\right), \\
\left(a_{t+1}\left(h^t\right) - \underline{a}\right) &\left(\tilde{u}_c\left(h^t\right) - \beta\left(1+\bar{r}_{t+1}\right) \sum_{h^{t+1}} \Pi\left(h^{t+1} \mid h^t\right) \tilde{u}_c\left(h^{t+1}\right)\right) = 0, \\
a_{t+1}\left(h^t\right) &\geq \underline{a}.
\end{aligned}$$

Factor prices are set competitively: $r_t = f_K(K_t, N_t),\ w_t = f_N(K_t, N_t)$.Government budget constraint holds:

$$G_t + T_t + \left(1+\bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t = F\left(K_t, N_t\right) + \tau^c C_t + B_{t+1}.$$

Markets clear:

$$\begin{aligned}
\tilde{N}_t &= \sum_{h^t} \Pi\left(h^t\right) e_t n_t\left(h^t\right), \\
\tilde{K}_t &= \sum_{h^{t-1}} \Pi\left(h^{t-1}\right) a_t\left(h^{t-1}\right) - B_t, \\
\tilde{C}_t &= \sum_{h^t} \Pi\left(h^t\right) \tilde{c}_t\left(h^t\right).
\end{aligned}$$

5.2 Ramsey Problem

Given K_0, B_0, τ_0^k, τ_0^n, T_0, $\Pi\left(h^0\right)$ and a welfare function W, the **Ramsey problem** is to solve

$$\max_{\{\bar{w}_t, \bar{r}_t, T_t, B_{t+1}, a_{t+1}(h^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) u\left(c_t\left(h^t\right), n_t\left(h^t\right)\right)$$

subject to

$$\begin{aligned}
\tilde{u}_c\left(h^t\right) &\geq \beta\left(1+\bar{r}_{t+1}\right) \mathbb{E}\left[\tilde{u}_c\left(h^{t+1}\right) \mid h^t\right], \\
\left(a_{t+1}\left(h^t\right)-\underline{a}\right)&\left(n\tilde{u}_c\left(h^t\right)-\beta\left(1+\bar{r}_{t+1}\right) \mathbb{E}\left[\tilde{u}_c\left(h^{t+1}\right) \mid h^t\right]\right)=0, \\
a_{t+1}\left(h^t\right) &\geq \underline{a}, \\
F\left(\tilde{K}_t, \tilde{N}_t\right)+B_{t+1}+\tau^c \tilde{C}_t &= G_t+T_t+\left(1+\bar{r}_t\right) B_t+\bar{r}_t \tilde{K}_t+\bar{w}_t \tilde{N}_t .
\end{aligned}$$

5.2.1 Lagrangian

$$\begin{aligned}
\mathcal{L}=\sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \Big\{ & \tilde{u}\left(h^t\right)+\theta_{t+1}\left(h^t\right)\left[\tilde{u}_c\left(h^t\right)-\beta\left(1+\bar{r}_{t+1}\right) \mathbb{E}\left[\tilde{u}_c\left(h^{t+1}\right) \mid h^t\right]\right] \\
& -\eta_{t+1}\left(h^t\right)\left[\left(a_{t+1}\left(h^t\right)-\underline{a}\right)\left(\tilde{u}_c\left(h^t\right)-\beta\left(1+\bar{r}_{t+1}\right) \mathbb{E}\left[\tilde{u}_c\left(h^{t+1}\right) \mid h^t\right]\right)\right]\Big\} \\
& +\sum_{t=0}^{\infty} \beta^t \gamma_t\left[F\left(\tilde{K}_t, \tilde{N}_t\right)+B_{t+1}+\tau^c \tilde{C}_t-\left(G_t+T_t+\left(1+\bar{r}_t\right) B_t+\bar{r}_t \tilde{K}_t+\bar{w}_t \tilde{N}_t\right)\right]
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{L}=\sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \Big\{ & \tilde{u}\left(h^t\right)-\lambda_{t+1}\left(h^t\right)\left[\tilde{u}_c\left(h^t\right)-\beta\left(1+\bar{r}_{t+1}\right) \mathbb{E}\left[\tilde{u}_c\left(h^{t+1}\right) \mid h^t\right]\right]\Big\} \\
& +\sum_{t=0}^{\infty} \beta^t \gamma_t\left[F\left(\tilde{K}_t, \tilde{N}_t\right)+B_{t+1}+\tau^c \tilde{C}_t-\left(G_t+T_t+\left(1+\bar{r}_t\right) B_t+\bar{r}_t \tilde{K}_t+\bar{w}_t \tilde{N}_t\right)\right]
\end{aligned}$$

with

$$\lambda_{t+1}\left(h^t\right) \equiv \eta_{t+1}\left(h^t\right)\left(a_{t+1}\left(h^t\right)-\underline{a}\right)-\theta_{t+1}\left(h^t\right).$$

And setting

$$\lambda_0\left(h^{t-1}\right) \equiv 0,$$

we get

$$\mathcal{L}=\sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right)\left[\tilde{u}\left(h^t\right)+\left(\lambda_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right)-\lambda_{t+1}\left(h^t\right)\right) \tilde{u}_c\left(h^t\right)\right]$$

$$+\sum_{t=0}^{\infty}\beta^t\gamma_t\left[F\left(\tilde{K}_t,\tilde{N}_t\right)+B_{t+1}-\left(G_t+T_t+(1+\bar{r}_t)B_t+\bar{r}_t\tilde{K}_t+\bar{w}_t\tilde{N}_t-\tau_t^c\tilde{C}_t\right)\right]$$

5.3 Recursive Formulation

For each $h^0\in A\times E$, let $\Pi\left(h^0\right)$ denote the initial measure of households with that initial state. Let $\mu_0\left(\lambda_0,a_0,e_0\right)$ denote the trivial extension of this distribution to $\mathbb{R}\times A\times E$ space with all probability mass on $\lambda_0=0$.

Then, the objective of the social planner can be represented by $W\left(\mu_0,B_0\right)$ where W solves the following problem:

$$W(\mu,B)=\min_{\lambda'(\cdot)\geq 0}\max_{\bar{w},\bar{r},T,B',a'(\cdot)}\sum_e\int\tilde{u}(\cdot)+\left(\lambda'(\cdot)(1+\bar{r})-\lambda\right)\tilde{u}_c(\cdot)\,\mu(ds,e)+\beta W\left(\mu',B'\right)$$

subject to

$$a'(\cdot)\geq\underline{a},$$

$$F\left(\tilde{K},\tilde{N}\right)=G+T+(1+\bar{r})B+\bar{r}\tilde{K}+\bar{w}\tilde{N}-\tau^c\tilde{C}-B',$$

μ' is consistent with $\lambda'(\cdot)$ and $a'(\cdot)$, and

$$\tilde{c}=\frac{\bar{w}e\tilde{n}+(1+\bar{r})a+T-a'(\cdot)}{1+\tau^c},\ \ \tilde{n}=\left(\frac{\bar{w}_te_t\left(h^t\right)}{(1+\tau^c)\chi}\right)^{\varepsilon},\ \ \tilde{u}=u\left(\tilde{c},\tilde{n}\right),$$

$$\tilde{N}=\sum_e\int e\tilde{n}\left(h^t\right)\mu(ds,e),\ \tilde{C}_t=\sum_e\int\tilde{c}_t\mu(ds,e),\tilde{K}_t=\sum_e\int a\mu(ds,e)-B.$$

5.4 First Order Conditions (Ignoring $I_{\{a>0\}}$)

Defining the Lagrangian

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^t\sum_{h^t}\Pi\left(h^t\right)\left[\tilde{u}\left(h^t\right)+\left(\lambda_t\left(h^{t-1}\right)(1+\bar{r}_t)-\lambda_{t+1}\left(h^t\right)\right)\tilde{u}_c\left(h^t\right)\right]$$
$$+\sum_{t=0}^{\infty}\beta^t\gamma_t\left[F\left(\tilde{K}_t,\tilde{N}_t\right)+B_{t+1}+\tau^c\tilde{C}_t-\left(G_t+T_t+(1+\bar{r}_t)B_t+\bar{r}_t\tilde{K}_t+\bar{w}_t\tilde{N}_t\right)\right]$$

and using

$$\tilde{c}_t\left(h^t\right)=\frac{\bar{w}_te_t\left(h^t\right)n_t\left(h^t\right)+(1+\bar{r}_t)a_t\left(h^{t-1}\right)+T_t-a_{t+1}\left(h^t\right)}{1+\tau^c},$$

$$\tilde{n}_t\left(h^t\right) = \left(\frac{\bar{w}_t e_t\left(h^t\right)}{\left(1+\tau^c\right)\chi}\right)^{\varepsilon},$$
$$\tilde{N}_t = \sum_{h^t} \Pi\left(h^t\right) e_t n_t\left(h^t\right),$$
$$\tilde{K}_t = \sum_{h^{t-1}} \Pi\left(h^{t-1}\right) a_t\left(h^{t-1}\right) - B_t,$$
$$\tilde{C}_t = \sum_{h^t} \Pi\left(h^t\right) \tilde{c}_t\left(h^t\right).$$

we obtain

$$[B_{t+1}] : \beta^t \gamma_t = \beta^{t+1}\gamma_{t+1}\left(1 + F_K\left(\tilde{K}_{t+1}, \tilde{N}_{t+1}\right)\right)$$
$$[T_t] : \sum_{h^t} \Pi\left(h^t\right)\left[\tilde{u}_c\left(h^t\right) + \left(\lambda_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cc}\left(h^t\right)\right] = \gamma_t$$
$$[\bar{r}_t] : \sum_{h^t} \Pi\left(h^t\right)\left[\tilde{u}_c\left(h^t\right) + \lambda_t\left(h^{t-1}\right)\tilde{u}_c\left(h^t\right) + \left(\lambda_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cc}\left(h^t\right)\right]\frac{a_t\left(h^{t-1}\right)}{1+\tau^c}$$
$$= \gamma_t\left[A_t - \tau^c \sum_{h^t}\Pi\left(h^t\right)\frac{a_t\left(h^{t-1}\right)}{1+\tau^c}\right]$$
$$[\bar{w}_t] : \sum_{h^t} \Pi\left(h^t\right)\left[\tilde{u}_c\left(h^t\right) + \left(\lambda_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cc}\left(h^t\right)\right]\left(\frac{e_t\left(h^t\right)\tilde{n}_t\left(h^t\right)}{1+\tau^c} + \frac{\bar{w}_t e_t\left(h^t\right)}{1+\tau^c}\varepsilon\frac{\tilde{n}_t\left(h^t\right)}{\bar{w}_t}\right)$$
$$+ \sum_{h^t} \Pi\left(h^t\right)\left[\tilde{u}_n\left(h^t\right) + \left(\lambda_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cn}\left(h^t\right)\right]\varepsilon\frac{\tilde{n}_t\left(h^t\right)}{\bar{w}_t}$$
$$+ \gamma_t\left[F_N\left(\tilde{K}_t, \tilde{N}_t\right)\varepsilon\frac{\tilde{N}_t}{\bar{w}_t} + \tau^c\sum_{h^t}\Pi\left(h^t\right)\left(\frac{e_t\left(h^t\right)\tilde{n}_t\left(h^t\right)}{1+\tau^c} + \frac{\bar{w}_t e_t\left(h^t\right)}{1+\tau^c}\varepsilon\frac{\tilde{n}_t\left(h^t\right)}{\bar{w}_t}\right) - \left(\tilde{N}_t + \bar{w}_t\varepsilon\frac{\tilde{N}_t}{\bar{w}_t}\right)\right] = 0$$
$$[a_{t+1}] : -\beta^t\left[\tilde{u}_c\left(h^t\right)\frac{1}{1+\tau^c} + \left(\lambda_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cc}\left(h^t\right)\frac{1}{1+\tau^c}\right] - \beta^t\gamma_t\tau^c\frac{1}{1+\tau^c}$$
$$+ \beta^{t+1}\sum_{h^{t+1}}\Pi\left(h^{t+1} \mid h^t\right)\left[\tilde{u}_c\left(h^{t+1}\right)\frac{1+\bar{r}_{t+1}}{1+\tau^c} + \left(\lambda_{t+1}\left(h^t\right)\left(1+\bar{r}_{t+1}\right) - \lambda_{t+2}\left(h^{t+1}\right)\right)\tilde{u}_{cc}\left(h^{t+1}\right)\frac{1+\bar{r}_{t+1}}{1+\tau^c}\right]$$
$$+ \beta^{t+1}\gamma_{t+1}\left[F_K\left(\tilde{K}_{t+1}, \tilde{N}_{t+1}\right) - \bar{r}_{t+1} + \tau^c\frac{1+\bar{r}_{t+1}}{1+\tau^c}\right] = 0$$

5.4.1 In Stationary Equilibrium

$$[B] : 1 = \beta\left(1 + F_K\left(K, N\right)\right) \tag{5.1}$$
$$[a'] : \left(\frac{\lambda}{\gamma}\left(1+\bar{r}\right) - \frac{\lambda'}{\gamma}\right)\tilde{u}_{cc} = \beta\left(1+\bar{r}\right)\mathbb{E}\left[\left(\frac{\lambda'}{\gamma}\left(1+\bar{r}'\right) - \frac{\lambda''}{\gamma}\right)\tilde{u}'_{cc} \mid e\right] + \left(\beta\left(1+\bar{r}\right) - 1\right)\tau^c + \beta\left(1+\tau^c\right)\tau^k r \tag{5.2}$$

$$[T] : \gamma = \sum_e \int \int \left(\tilde{u}_c + \left(\lambda \left(1 + \bar{r} \right) - \lambda' \right) \tilde{u}_{cc} \right) p \left(\lambda, a, e \right) da d\lambda \tag{5.3}$$

$$[\bar{r}] : A = \sum_e \int \int \left\{ \left[\frac{\tilde{u}_c}{\gamma} + \left(\frac{\lambda}{\gamma} \left(1 + \bar{r} \right) - \frac{\lambda'}{\gamma} \right) \tilde{u}_{cc} \right] a + \frac{\lambda}{\gamma} \left(1 + \tau^c \right) \tilde{u}_c \right\} p \left(\lambda, a, e \right) da d\lambda \tag{5.4}$$

$$[\bar{w}] : \tilde{N} = \sum_e \int \int \left[\left(\frac{\tilde{u}_c}{\gamma} + \left(\frac{\lambda}{\gamma} \left(1 + \bar{r} \right) - \frac{\lambda'}{\gamma} \right) \tilde{u}_{cc} \right) en \right] p \left(\lambda, a, e \right) da d\lambda + \left(\frac{\tau^c + \tau^n}{1 - \tau^n} \right) \varepsilon \tilde{N} \tag{5.5}$$

Define $q' \left(q, a, e \right) = \lambda' \left(q, a, e \right) / \gamma$ and notice that it follows from equation (5.2) that

$$\begin{aligned} \left(q \left(1 + \bar{r} \right) - q' \left(q, a, e \right) \right) \tilde{u}_{cc} = & \beta \left(1 + \bar{r} \right) \mathbb{E} \left[\left(q' \left(q, a, e \right) \left(1 + \bar{r} \right) - q'' \left(q', a', e' \right) \right) \tilde{u}'_{cc} \mid e \right] \\ & + \left(\beta \left(1 + \bar{r} \right) - 1 \right) \tau^c + \beta \left(1 + \tau^c \right) \tau^k r. \end{aligned}$$

Hence, $q' \left(q, a, e \right)$ is linear in q. Accordingly, define $b_0 \left(a, e \right)$ and $b_1 \left(a, e \right)$ such that

$$q' \left(q, a, e \right) = b_0 \left(a, e \right) + b_1 \left(a, e \right) q$$

and it follows that

$$b_1 \left(a, e \right) = \frac{\left(1 + \bar{r} \right) \tilde{u}_{cc}}{\tilde{u}_{cc} + \beta \left(1 + \bar{r} \right) \mathbb{E} \left[\left(\left(1 + \bar{r} \right) - b'_1 \left(a', e' \right) \right) \tilde{u}'_{cc} \mid e \right]} \tag{5.6}$$

$$b_0 \left(a, e \right) = \frac{\beta \left(1 + \bar{r} \right) \mathbb{E} \left[b'_0 \left(a', e' \right) \tilde{u}'_{cc} \mid e \right] - \left(\beta \left(1 + \bar{r} \right) - 1 \right) \tau^c - \beta \left(1 + \tau^c \right) \tau^k r}{\tilde{u}_{cc} + \beta \left(1 + \bar{r} \right) \mathbb{E} \left[\left(\left(1 + \bar{r} \right) - b'_1 \left(a', e' \right) \right) \tilde{u}'_{cc} \mid e \right]} \tag{5.7}$$

5.4.2 Algorithm

1. Guess τ^k, τ^n, and B.

2. Compute the associated stationary equilibrium. In particular, obtain the policy functions $a' \left(a, e \right)$ and $n \left(a, e \right)$ (which give $\tilde{u}_c \left(a, e \right)$, $\tilde{u}_{cc} \left(a, e \right)$, and $\tilde{u}'_{cc} \left(a, e \right)$), prices r, and w, and aggregates $\tilde{N}$, $\tilde{K}$, $\tilde{C}$, and A.

3. Guess $b_1 \left(a, e \right)$ and use equations (5.6) and (5.7) to obtain $q' \left(q, a, e \right)$.

4. Use equation (5.3) to find γ, i.e.

$$\gamma = \frac{\sum_e \int \int \tilde{u}_c p \left(\lambda, a, e \right) da d\lambda}{1 - \sum_e \int \int \left(\left(\frac{\lambda}{\gamma} \left(1 + \bar{r} \right) - \frac{\lambda'}{\gamma} \right) \tilde{u}_{cc} \right) p \left(\lambda, a, e \right) da d\lambda},$$

   and use it to find $\lambda' \left(\lambda, a, e \right)$.

5. Use $\lambda' \left(\lambda, a, e \right)$, $a' \left(a, e \right)$, and the Markov matrix for e to find $p \left(\lambda, a, e \right)$.

6. Use equations (5.1), (5.4), and (5.5) to solve for new τ^k, τ^n, and B, i.e.

$$\tau^k = 1 - \frac{A - \sum_e \int\int \left\{ \left[\frac{\tilde{u}_c}{\gamma} + \left(\frac{\lambda}{\gamma} - \frac{\lambda'}{\gamma} \right) \tilde{u}_{cc} \right] a + \frac{\lambda}{\gamma} (1+\tau^c) \tilde{u}_c \right\} p(\lambda, a, e)\, da d\lambda}{r \sum_e \int\int \left\{ \left[\frac{\lambda}{\gamma} \tilde{u}_{cc} \right] a + \frac{\lambda}{\gamma} (1+\tau^c) \tilde{u}_c \right\} p(\lambda, a, e)\, da d\lambda}$$

$$\tau^n = \frac{\tilde{N} - \sum_e \int\int \left[\left(\frac{\tilde{u}_c}{\gamma} + \left(\frac{\lambda}{\gamma}(1+\bar{r}) - \frac{\lambda'}{\gamma} \right) \tilde{u}_{cc} \right) en \right] p(\lambda, a, e)\, da d\lambda - \tau^c \varepsilon \tilde{N}}{\varepsilon \tilde{N} + \tilde{N} - \sum_e \int\int \left[\left(\frac{\tilde{u}_c}{\gamma} + \left(\frac{\lambda}{\gamma}(1+\bar{r}) - \frac{\lambda'}{\gamma} \right) \tilde{u}_{cc} \right) en \right] p(\lambda, a, e)\, da d\lambda}$$

$$B = A - \left(\frac{1 - \beta(1-\delta)}{\alpha \beta \tilde{N}^{1-\alpha}} \right)^{\frac{1}{\alpha - 1}}$$

7. Go back to step 2 if the new τ^k, τ^n, and B differs enough from the previous ones.

5.5 First Order Conditions (taking care of $I_{\{a>0\}}$)

Defining the Lagrangian

$$\begin{aligned}
\mathcal{L} = & \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi(h^t) \left[\tilde{u}(h^t) + \left(\lambda_t(h^{t-1}) \left(1 + I_{\{a>0\}} \bar{r}_t + I_{\{a<0\}} F_K\left(\tilde{K}_t, \tilde{N}_t\right) \right) - \lambda_{t+1}(h^t) \right) \tilde{u}_c(h^t) \right] \\
& + \sum_{t=0}^{\infty} \beta^t \gamma_t \Big[F\left(\tilde{K}_t, \tilde{N}_t\right) + B_{t+1} + \tau^c \tilde{C}_t + \left(F_K\left(\tilde{K}_t, \tilde{N}_t\right) - \bar{r}_t \right) \left(\hat{A}_t - A_t \right) \\
& - \left(G_t + T_t + (1+\bar{r}_t) B_t + \bar{r}_t \tilde{K}_t + \bar{w}_t \tilde{N}_t \right) \Big]
\end{aligned}$$

and using

$$\begin{aligned}
\tilde{c}_t(h^t) &= \frac{\bar{w}_t e_t(h^t) n_t(h^t) + \left(1 + I_{\{a>0\}} \bar{r}_t + I_{\{a<0\}} F_K\left(\tilde{K}_t, \tilde{N}_t\right) \right) a_t(h^{t-1}) + T_t - a_{t+1}(h^t)}{1+\tau^c}, \\
\tilde{n}_t(h^t) &= \left(\frac{\bar{w}_t e_t(h^t)}{(1+\tau^c)\chi} \right)^{\varepsilon}, \\
\tilde{N}_t &= \sum_{h^t} \Pi(h^t) e_t n_t(h^t), \\
\tilde{K}_t &= \sum_{h^{t-1}} \Pi(h^{t-1}) a_t(h^{t-1}) - B_t, \\
\tilde{C}_t &= \sum_{h^t} \Pi(h^t) \tilde{c}_t(h^t).
\end{aligned}$$

we obtain

$$[B_{t+1}]:\gamma_t=\beta\sum_{h^{t+1}}\Pi\left(h^{t+1}\right)\Bigg[\left(\tilde{u}_c\left(h^{t+1}\right)+\left(\lambda_{t+1}\left(h^{t+1-1}\right)\left(1+I_{\{a>0\}}\bar{r}_{t+1}+I_{\{a<0\}}F_K\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\right)-\lambda_{t+2}\left(h^{t+1}\right)\right)\tilde{u}_{cc}\left(h^{t+1}\right)\right)$$

$$\frac{I_{\{a<0\}}F_{KK}\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)a_{t+1}\left(h^t\right)}{1+\tau^c}+\lambda_{t+1}\left(h^{t+1-1}\right)\left(I_{\{a<0\}}F_{KK}\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\right)\tilde{u}_c\left(h^{t+1}\right)\Bigg]$$

$$+\beta\gamma_{t+1}\left[1+F_K\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)+\tau^c\frac{F_{KK}\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\left(A_{t+1}-\hat{A}_{t+1}\right)}{1+\tau^c}+F_{KK}\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\left(\hat{A}_{t+1}-A_{t+1}\right)\right]$$

$$[T_t]:\sum_{h^t}\Pi\left(h^t\right)\left[\tilde{u}_c\left(h^t\right)+\left(\lambda_t\left(h^{t-1}\right)\left(1+I_{\{a>0\}}\bar{r}_t+I_{\{a<0\}}F_K\left(\tilde{K}_t,\tilde{N}_t\right)\right)-\lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cc}\left(h^t\right)\right]=\gamma_t$$

$$[\bar{r}_t]:\sum_{h^t}\Pi\left(h^t\right)\Bigg[\left(\tilde{u}_c\left(h^t\right)+\left(\lambda_t\left(h^{t-1}\right)\left(1+I_{\{a>0\}}\bar{r}_t+I_{\{a<0\}}F_K\left(\tilde{K}_t,\tilde{N}_t\right)\right)-\lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cc}\left(h^t\right)\right)I_{\{a>0\}}a_t\left(h^{t-1}\right)$$

$$+\lambda_t\left(h^{t-1}\right)I_{\{a>0\}}\tilde{u}_c\left(h^t\right)\Bigg]=\frac{\gamma_t\hat{A}_t}{1+\tau^c}$$

$$[\bar{w}_t]:\sum_{h^t}\Pi\left(h^t\right)\left[\tilde{u}_c\left(h^t\right)+\left(\lambda_t\left(h^{t-1}\right)\left(1+I_{\{a>0\}}\bar{r}_t+I_{\{a<0\}}F_K\left(\tilde{K}_t,\tilde{N}_t\right)\right)-\lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cc}\left(h^t\right)\right]\left(\frac{e_t\left(h^t\right)\tilde{n}_t\left(h^t\right)}{1+\tau^c}+\frac{\bar{w}_te_t\left(h^t\right)}{1+\tau^c}\varepsilon\frac{\tilde{n}_t\left(h^t\right)}{\bar{w}_t}\right)$$

$$+\sum_{h^t}\Pi\left(h^t\right)\left[\tilde{u}_n\left(h^t\right)+\left(\lambda_t\left(h^{t-1}\right)\left(1+I_{\{a>0\}}\bar{r}_t+I_{\{a<0\}}F_K\left(\tilde{K}_t,\tilde{N}_t\right)\right)-\lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cn}\left(h^t\right)\right]\varepsilon\frac{\tilde{n}_t\left(h^t\right)}{\bar{w}_t}$$

$$+\sum_{h^t}\Pi\left(h^t\right)\left[\lambda_t\left(h^{t-1}\right)\left(I_{\{a<0\}}F_{KN}\left(\tilde{K}_t,\tilde{N}_t\right)\right)\tilde{u}_c\left(h^t\right)\right]\varepsilon\frac{\tilde{N}_t}{\bar{w}_t}+\gamma_t\Bigg[F_N\left(\tilde{K}_t,\tilde{N}_t\right)\varepsilon\frac{\tilde{N}_t}{\bar{w}_t}$$

$$+\tau^c\sum_{h^t}\Pi\left(h^t\right)\left(\frac{e_t\left(h^t\right)\tilde{n}_t\left(h^t\right)}{1+\tau^c}+\frac{\bar{w}_te_t\left(h^t\right)}{1+\tau^c}\varepsilon\frac{\tilde{n}_t\left(h^t\right)}{\bar{w}_t}+\frac{I_{\{a<0\}}F_{KN}\left(\tilde{K}_t,\tilde{N}_t\right)a_t\left(h^{t-1}\right)}{1+\tau^c}\varepsilon\frac{\tilde{N}_t}{\bar{w}_t}\right)$$

$$+F_{KN}\left(\tilde{K}_t,\tilde{N}_t\right)\left(\hat{A}_t-A_t\right)\varepsilon\frac{\tilde{N}_t}{\bar{w}_t}-\left(\tilde{N}_t+\bar{w}_t\varepsilon\frac{\tilde{N}_t}{\bar{w}_t}\right)\Bigg]$$

$$[a_{t+1}]:\Pi\left(h^t\right)\left[-\tilde{u}_c\left(h^t\right)\frac{1}{1+\tau^c}-\left(\lambda_t\left(h^{t-1}\right)\left(1+I_{\{a>0\}}\bar{r}_t+I_{\{a<0\}}F_K\left(\tilde{K}_t,\tilde{N}_t\right)\right)-\lambda_{t+1}\left(h^t\right)\right)\tilde{u}_{cc}\left(h^t\right)\frac{1}{1+\tau^c}\right]-\gamma_t\tau^c\frac{1}{1+\tau^c}\Pi\left(h^t\right)$$

$$+\beta\sum_{h^{t+1}}\Pi\left(h^{t+1}\right)\left[\tilde{u}_c\left(h^{t+1}\right)+\left(\lambda_{t+1}\left(h^{t+1-1}\right)\left(1+I_{\{a>0\}}\bar{r}_{t+1}+I_{\{a<0\}}F_K\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\right)-\lambda_{t+2}\left(h^{t+1}\right)\right)\tilde{u}_{cc}\left(h^{t+1}\right)\right]$$

$$\frac{\left(1+I_{\{a>0\}}\bar{r}_{t+1}+I_{\{a<0\}}F_K\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\right)+I_{\{a<0\}}F_{KK}\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\Pi\left(h^t\right)a_{t+1}\left(h^t\right)}{1+\tau^c}$$

$$+\beta\sum_{h^{t+1}}\Pi\left(h^{t+1}\right)\left[\lambda_{t+1}\left(h^{t+1-1}\right)I_{\{a<0\}}F_{KK}\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\Pi\left(h^t\right)\tilde{u}_c\left(h^{t+1}\right)\right]+\beta\gamma_{t+1}\Bigg[F_K\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\Pi\left(h^t\right)$$

$$+\tau^c\frac{\left(1+I_{\{a>0\}}\bar{r}_{t+1}+I_{\{a<0\}}F_K\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\right)+I_{\{a<0\}}F_{KK}\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\Pi\left(h^t\right)a_{t+1}\left(h^t\right)}{1+\tau^c}\Pi\left(h^t\right)$$

$$+\Pi\left(h^t\right)F_{KK}\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)\left(\hat{A}_{t+1}-A_{t+1}\right)+\left(F_K\left(\tilde{K}_{t+1},\tilde{N}_{t+1}\right)-\bar{r}_{t+1}\right)\left(I_{\{a>0\}}-1\right)\Pi\left(h^t\right)-\bar{r}_{t+1}\Pi\left(h^t\right)\Bigg]$$

5.5.1 Stationary Equilibrium

$$[B]:\gamma=\beta\sum_e\int\int\left\{\left[\left(\tilde{u}_c'+\left(\lambda'\left(1+I_{\{a>0\}}'\bar{r}+I_{\{a<0\}}'F_K\right)-\lambda''\right)\tilde{u}_{cc}'\right)\frac{I_{\{a<0\}}'a'}{1+\tau^c}+\lambda'I_{\{a<0\}}'\tilde{u}_c'\right]F_{KK}\right\}dad\lambda$$

$$+\beta\gamma\left[1+F_K+\frac{1}{1+\tau^c}F_{KK}\left(\hat{A}-A\right)\right]$$

$$[a']:\frac{\tilde{u}_c}{\gamma}+\left(\frac{\lambda}{\gamma}\left(1+I_{\{a>0\}}\bar{r}+I_{\{a<0\}}F_K\right)-\frac{\lambda'}{\gamma}\right)\tilde{u}_{cc}+\tau^c=\beta\Big[\left(1+\tau^c\right)I_{\{a>0\}}'\left(F_K-\bar{r}\right)$$

$$
\begin{aligned}
&+\tau^{c}\left(\left(1+I'_{\{a>0\}}\bar{r}+I'_{\{a<0\}}F_{K}\right)+\mu\left(a,e\right)I'_{\{a<0\}}F_{KK}a'\right)+\left(1+\tau^{c}\right)F_{KK}\left(\hat{A}-A\right)\Big]\\
&+\beta\mathbb{E}\Big[\left(\frac{\tilde{u}'_{c}}{\gamma}+\left(\frac{\lambda'}{\gamma}\left(1+I'_{\{a>0\}}\bar{r}+I'_{\{a<0\}}F_{K}\right)-\frac{\lambda''}{\gamma}\right)\tilde{u}'_{cc}\right)\left(\left(1+I'_{\{a>0\}}\bar{r}+I'_{\{a<0\}}F_{K}\right)+\mu\left(a,e\right)I'_{\{a<0\}}F_{KK}a'\right)\\
&+\left(1+\tau^{c}\right)\mu\left(a,e\right)\frac{\lambda'}{\gamma}I'_{\{a<0\}}F_{KK}\tilde{u}'_{c}\mid e\Big]
\end{aligned}
$$

$$
[T]:\gamma=\sum_{e}\int\int\left(\tilde{u}_{c}+\left(\lambda\left(1+I_{\{a>0\}}\bar{r}+I_{\{a<0\}}F_{K}\right)-\lambda'\right)\tilde{u}_{cc}\right)p\left(a,\lambda,e\right)dad\lambda
$$

$$
[\bar{r}]:\hat{A}=\sum_{e}\int\int\left\{\left(\frac{\tilde{u}_{c}}{\gamma}+\left(\frac{\lambda}{\gamma}\left(1+I_{\{a>0\}}\bar{r}+I_{\{a<0\}}F_{K}\right)-\frac{\lambda'}{\gamma}\right)\tilde{u}_{cc}\right)I_{\{a>0\}}a+\left(1+\tau^{c}\right)\frac{\lambda}{\gamma}I_{\{a>0\}}\tilde{u}_{c}\right\}p\left(a,\lambda,e\right)dad\lambda
$$

$$
\begin{aligned}
[\bar{w}]:\tilde{N}=&\sum_{e}\int\int\left\{\left[\frac{\tilde{u}_{c}}{\gamma}+\left(\frac{\lambda}{\gamma}\left(1+I_{\{a>0\}}\bar{r}+I_{\{a<0\}}F_{K}\right)-\frac{\lambda'}{\gamma}\right)\tilde{u}_{cc}\right]e\tilde{n}+\left(1+\tau^{c}\right)\frac{\lambda}{\gamma}\left(I_{\{a<0\}}\frac{F_{KN}}{\bar{w}}\right)\tilde{u}_{c}\varepsilon\tilde{N}\right\}p\left(a,\lambda,e\right)dad\lambda\\
&+\left[\left(\frac{\tau^{c}+\tau^{n}}{1-\tau^{n}}\right)+\frac{F_{KN}}{\bar{w}}\left(\hat{A}-A\right)\right]\varepsilon\tilde{N}
\end{aligned}
$$

where we use the fact that for the GHH utility function the following equations hold

$$
\tilde{u}_{n}=-\tilde{u}_{c}\chi n^{\frac{1}{\varepsilon}},\quad \tilde{u}_{cn}=-\tilde{u}_{cc}\chi n^{\frac{1}{\varepsilon}},\quad \frac{\partial n_{t}\left(h^{t}\right)}{\partial\bar{w}_{t}}=\varepsilon\frac{\tilde{n}_{t}\left(h^{t}\right)}{\bar{w}_{t}},\quad \text{and}\frac{\partial N_{t}}{\partial\bar{w}_{t}}=\varepsilon\frac{N_{t}}{\bar{w}_{t}}.
$$

5.5.2 Algorithm

The algorithm is exactly analogous to the one described in Section 5.4.2, except for the fact that steps 3 and 4 must be done concomitantly.

5.6 Results

Table 1 compares the long-run optimal fiscal policy we obtain using our solution method (Benchmark) with the one we obtain following the algorithm described above (Alternative). It is important to notice that the results from this alternative algorithm are sensitive to the number and placement of the grid points for q (which endogenously imply grid points for λ). The results in Table 1 were obtained with a grid of 1500 points logarithmically spaced between 0 and 10000 (the upper bound had to be sequentially increased until it was no longer hit).

Table 1: Long-run Optimal Taxes: Comparison

	τ^h	τ^k	T/Y	B/Y	K/Y	N	r	w
Benchmark	12.6	45.1	3.4	-15.1	2.819	0.386	3.52	1.170
Alternative	15.5	46.3	5.3	-29.7	2.796	0.375	3.63	1.164

Notes: The values of τ^h, τ^k, T/Y, B/Y, and r are in percentage points.

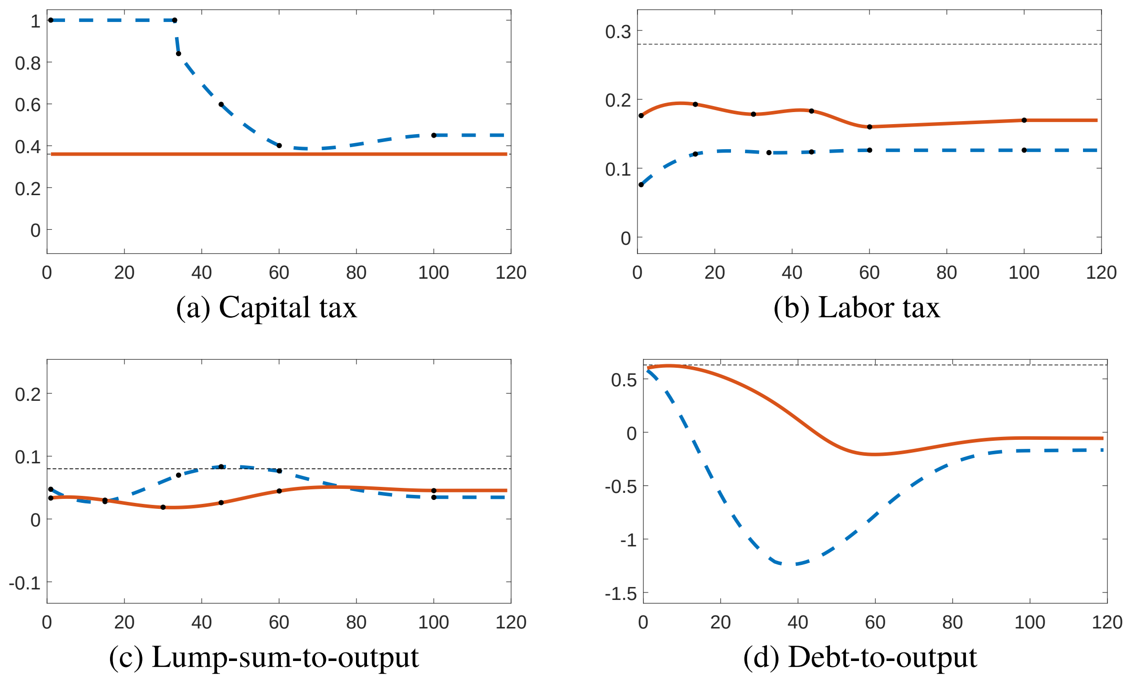


Figure 2: Optimal Fiscal Policy: Fixed Capital Taxes

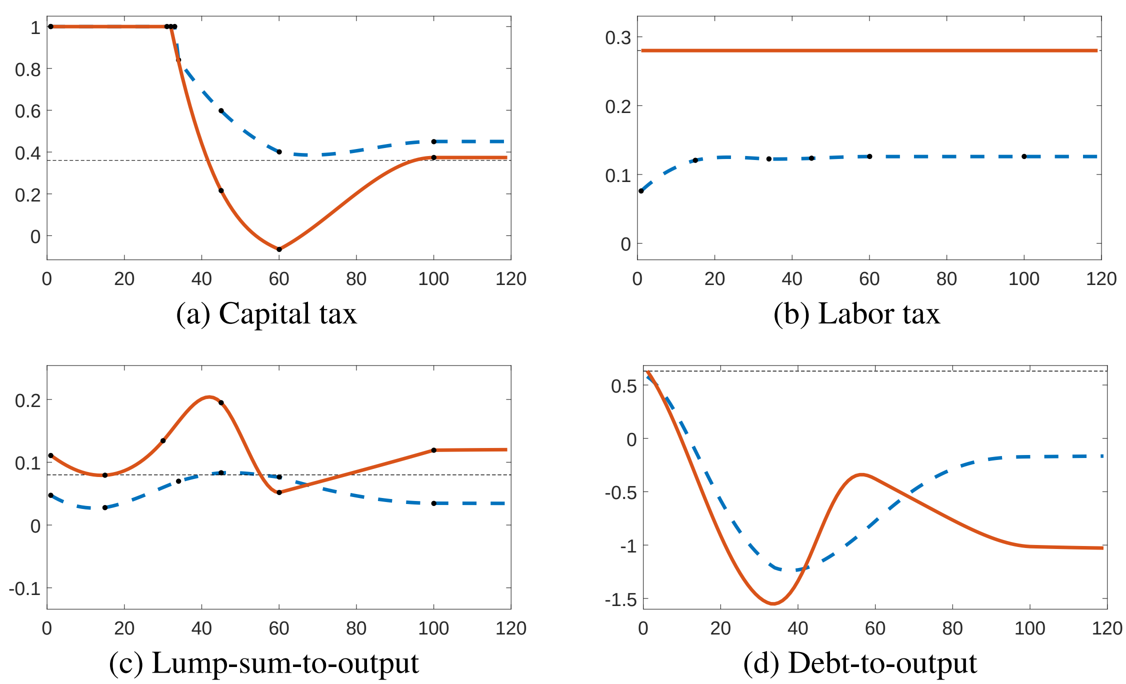
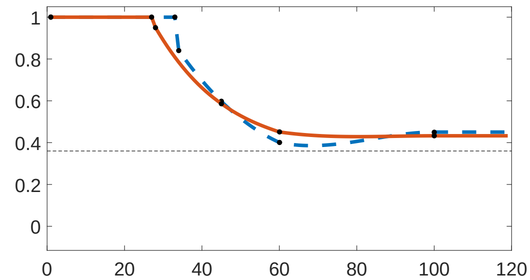
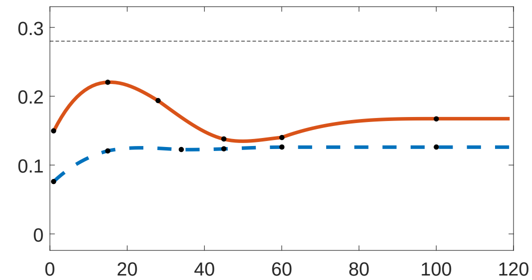


Figure 3: Optimal Fiscal Policy: Fixed Labor Taxes

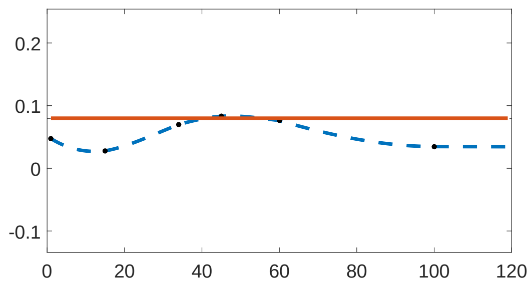
Notes: Dashed thin line: initial stationary equilibrium; Dashed thick line: optimal transition with unrestricted instruments (benchmark); Solid line: optimal transition with fixed instrument.



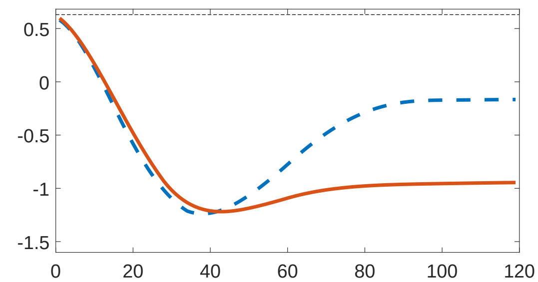
(a) Capital tax



(b) Labor tax

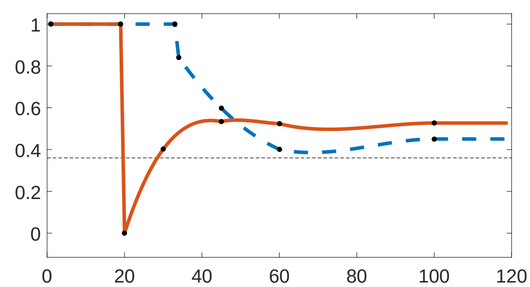


(c) Lump-sum-to-output

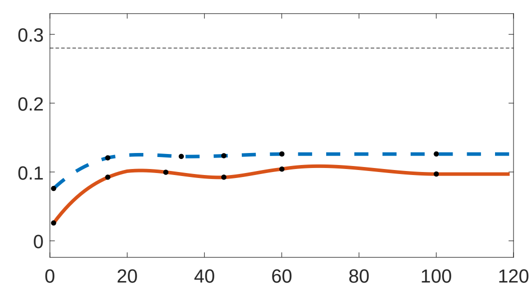


(d) Debt-to-output

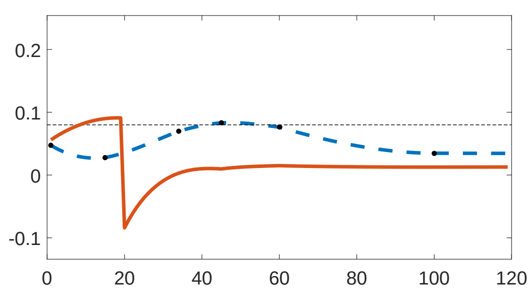
Figure 4: Optimal Fiscal Policy: Fixed Lump-Sum Transfers to Output



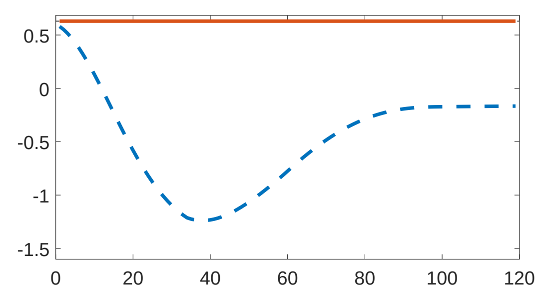
(a) Capital tax



(b) Labor tax



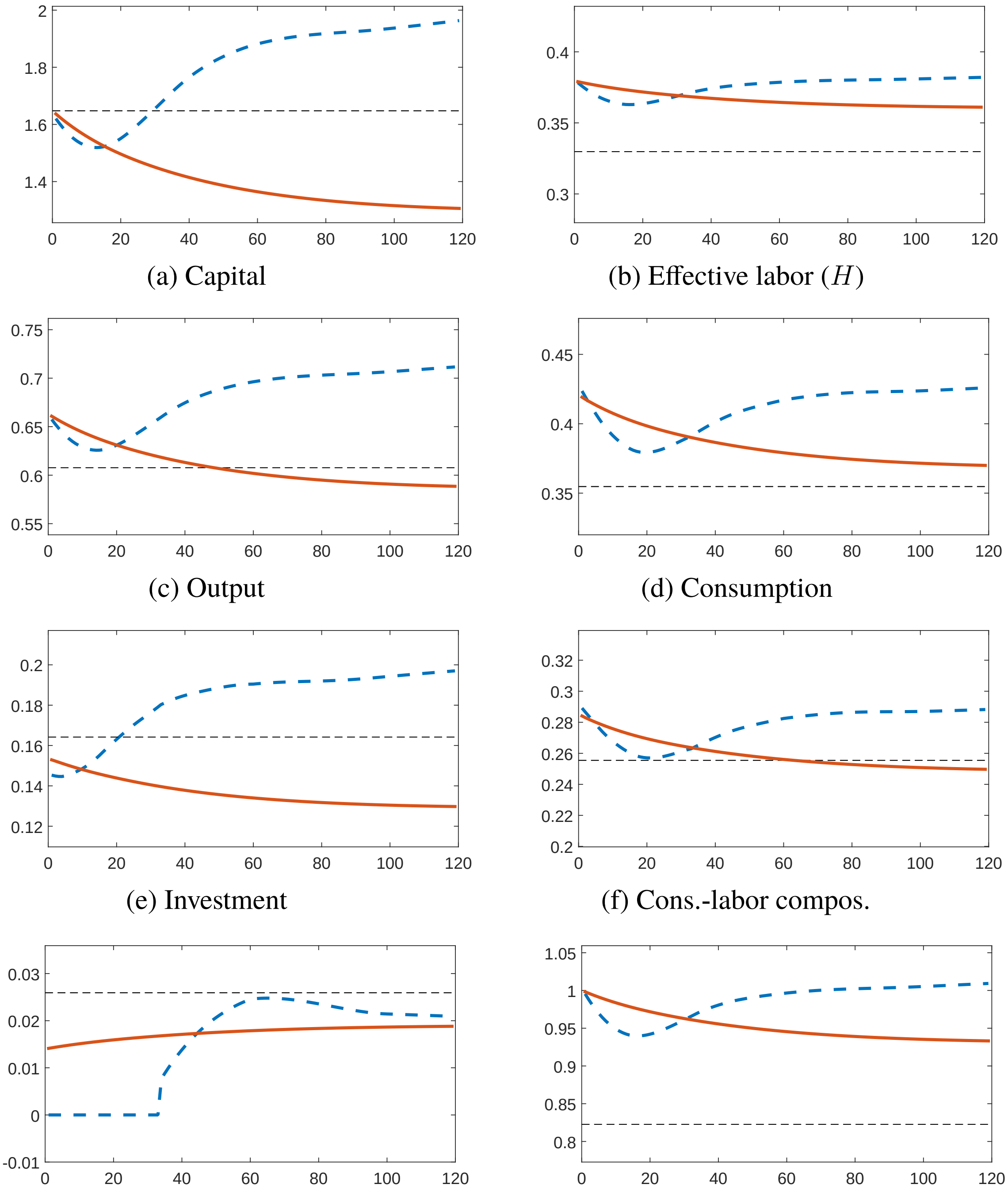
(c) Lump-sum-to-output



(d) Debt-to-output

Figure 5: Optimal Fiscal Policy: Fixed Debt-to-Output

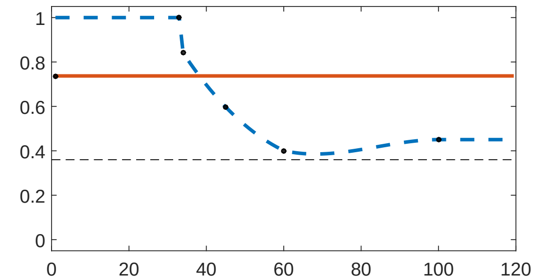
Notes: Dashed thin line: initial stationary equilibrium; Dashed thick line: optimal transition with unrestricted instruments (benchmark); Solid line: optimal transition with fixed instrument.



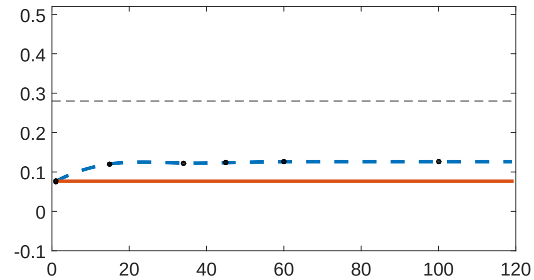
(a) Capital (b) Effective labor (H)
(c) Output (d) Consumption
(e) Investment (f) Cons.-labor compos.
(g) After-tax int. rates (h) After-tax wages

Figure 6: Aggregates: Constant Policy

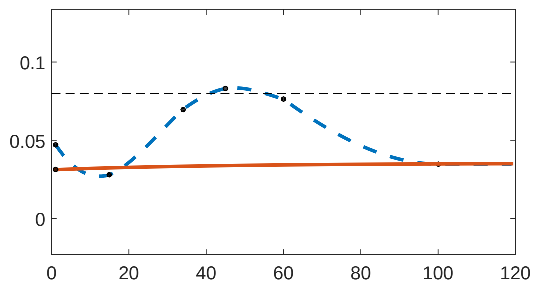
Notes: Dashed thin line: initial stationary equilibrium; Dashed thick line: optimal transition (benchmark); Solid line: optimal transition with constant taxes.



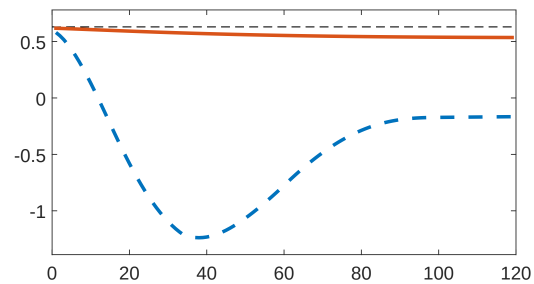
(a) Capital tax



(b) Labor tax



(c) Lump-sum-to-output



(d) Debt-to-output

Figure 7: Optimal Fiscal Policy: Constant Policy

Notes: Dashed thin line: initial stationary equilibrium; Dashed thick line: optimal transition (benchmark); Solid line: optimal transition with constant taxes; The black dots are the choice variables: the spline nodes and t^*, the point at which the capital tax leaves the upper bound.

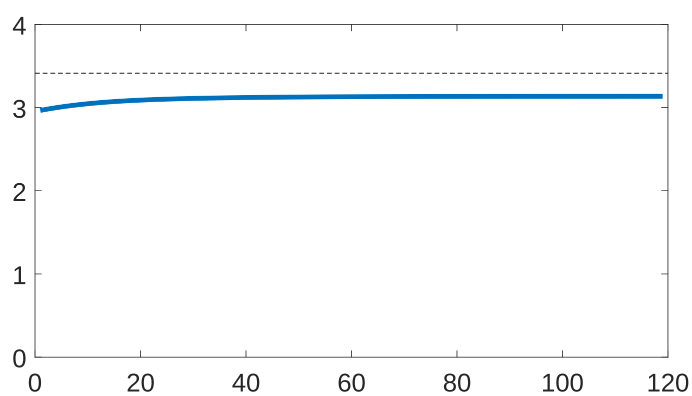


Figure 8: Economy 3: Θ_t

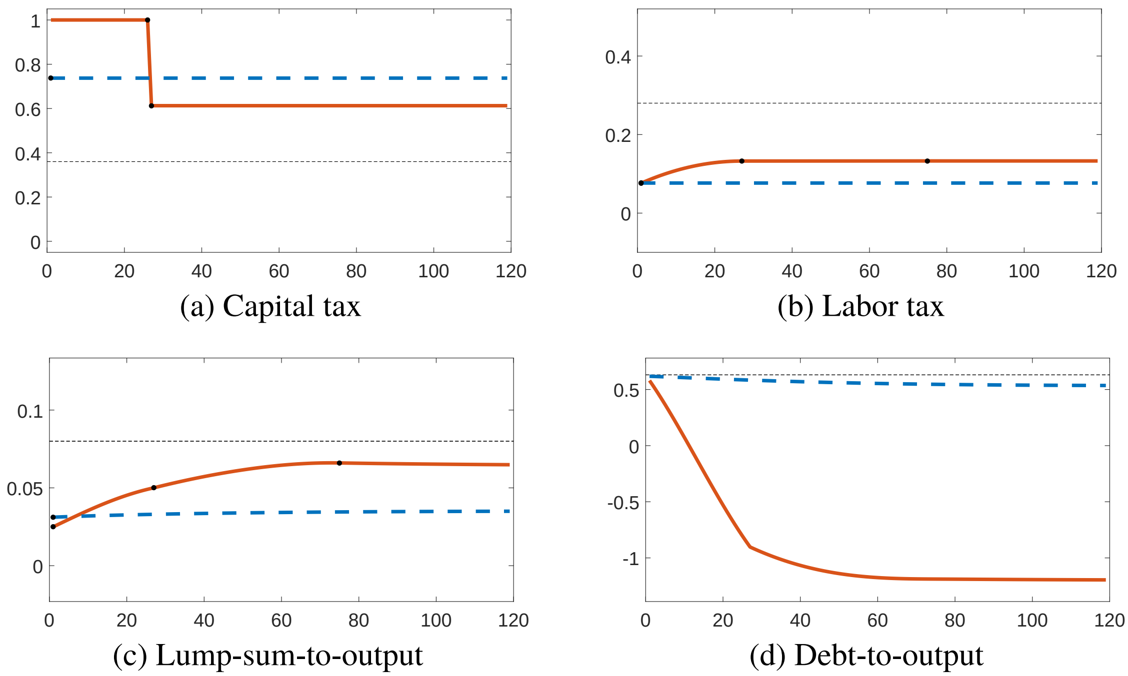


Figure 9: Number of Nodes: 4 to 6

Notes: Dashed line: optimal policy with 4 nodes; Solid line: optimal policy with 6 nodes.

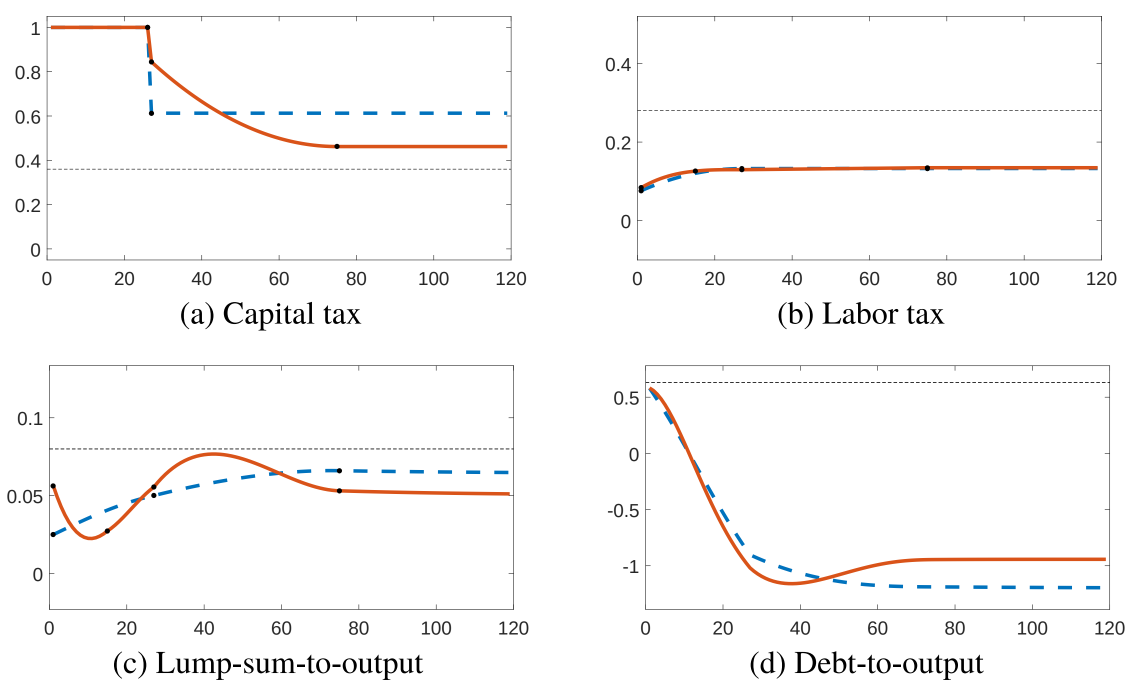
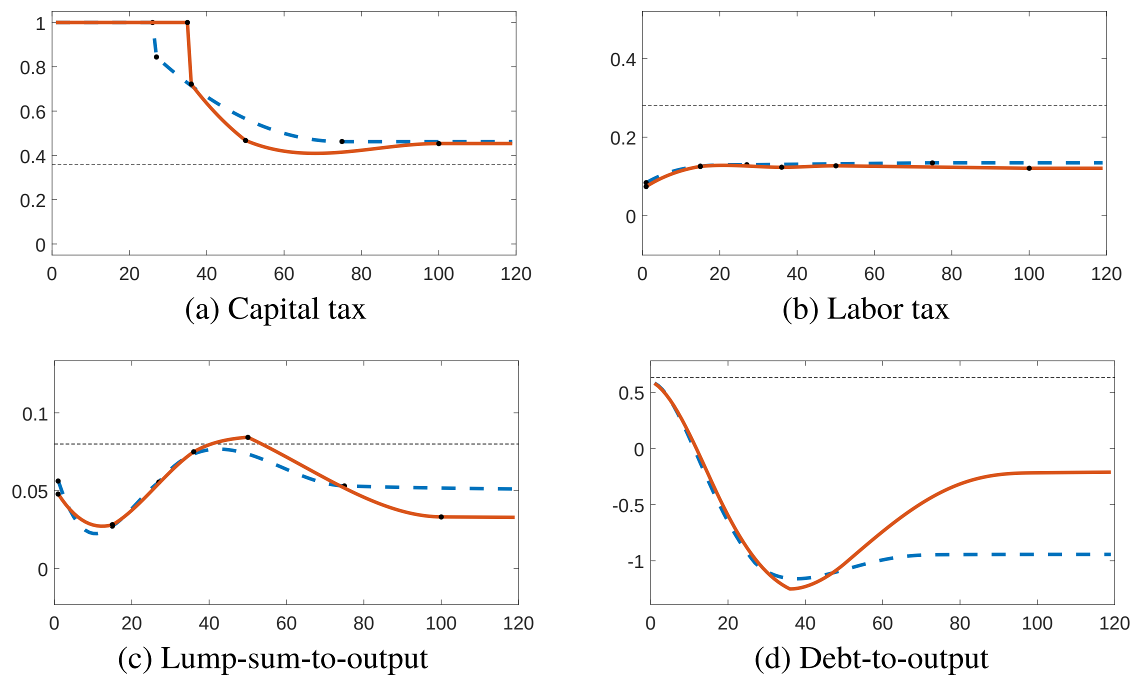


Figure 10: Number of Nodes: 6 to 9

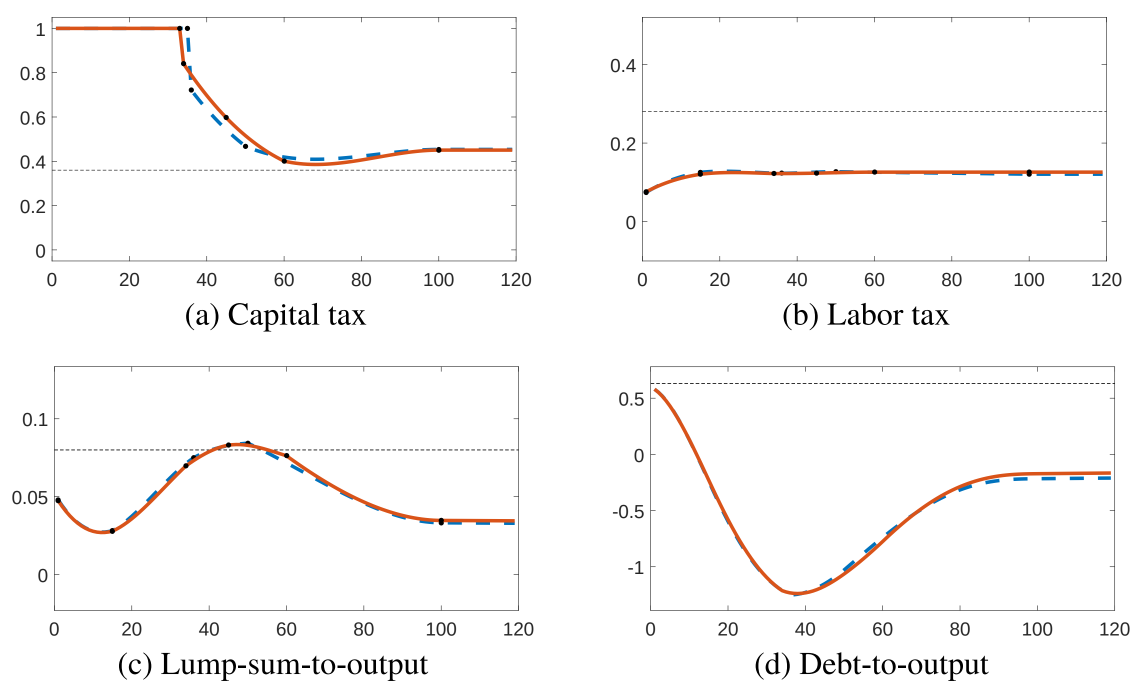
Notes: Dashed line: optimal policy with 6 nodes; Solid line: optimal policy with 9 nodes.



(a) Capital tax
(b) Labor tax
(c) Lump-sum-to-output
(d) Debt-to-output

Figure 11: Number of Nodes: 9 to 12

Notes: Dashed line: optimal policy with 9 nodes; Solid line: optimal policy with 12 nodes.



(a) Capital tax
(b) Labor tax
(c) Lump-sum-to-output
(d) Debt-to-output

Figure 12: Number of Nodes: 12 to 15

Notes: Dashed line: optimal policy with 12 nodes; Solid line: optimal policy with 15 nodes.

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